Definite Integrals and Area

PART-A : DEFINITE INTEGRALS Section - 1

1.1 Introduction

Consider a function $f(x)$ whose indefinite integral is $F(x) + C$.

i.e. $\int f(x)\,dx = F(x) + C$

Also consider the integral $\int_a^b f(x)\,dx$, which is known as a definite integral and $x = a, x = b$ are called the lower and upper limits of integration.

The relationship between the definite integral $\int_a^b f(x)\,dx$ and the indefinite integral $F(x)$ is :

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

This formula is known as Netwon–Leibnitz formula. This formula can used only if the function $f(x)$ is continuous at all points in the interval $[a, b]$.

Illustrating the Concepts **:**

Evaluate : **(i)** $\int_1^3 x^2 dx$ **(ii)** $\int_0^{\pi/2} \sin x\,dx$

(i) $\int_t^3 x^2\,dx = \left|\frac{x^3}{3}\right|_1^3 = \frac{1}{3}(3^3 - 1^3) = \frac{26}{3}$.

(ii) $\int_0^{\pi/2} \sin x\,dx = \left|-\cos x\right|_0^{\pi/2} = -(\cos \pi/2 - \cos 0) = 1.$

Illustration - 1 $\int_0^{\pi/2} \sin^3 x \cos x \, dx =$

(A) 1 (B) $\frac{1}{2}$ (C) $\frac{1}{3}$ (D) $\frac{1}{4}$

SOLUTION : (D)

Let $I = \int_0^{\pi/2} \sin^3 x \cos x \, dx$

Let $\sin x = t \Rightarrow \cos x \, dx = dt$

For $x = \frac{\pi}{2}, t = 1$ and for $x = 0$, $t = 0$.

$$\Rightarrow I = \int_0^1 t^3 \, dt = \left| \frac{t^4}{4} \right|_0^1 = \frac{1}{4}$$

Note : Whenever we use substitution in a definite integral, we have to change the limits corresponding to the change in the variable of the integration.

In this example we have applied Newton-Leibnitz formula to calculate the definite integral. Newton-Leibnitz formula is applicable here since $\sin^3 x \cos x$ (integrand) is a continuous function in the interval $[0, \pi/2]$.

PROPERTIES OF DEFINITE INTEGRALS — Section - 2

2.1 Basic Properties of Definite Integrals

PROPERTY - 1 :

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$$

Proof : Let $F(x)$ be indefinite integral of $f(x)$.
Using the Newton-Leibnitz formula,

$$\int_a^b f(x) dx \Rightarrow F(b) - F(a) \quad \ldots \text{(i)}$$

Also, $\int_a^c f(x)\, dx + \int_c^b f(x)\, dx = [F(c) - F(a)] + [F(b) - F(c)] = F(b) - F(a)$ **. . . (ii)**

From (i) and (ii), we get :

$$\int_a^b f(x)\,dx = \int_a^c f(x)dx = \int_c^b f(x)dx$$

PROPERTY - 2 :

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

Proof : Let $F(x)$ be indefinite integral of $f(x)$.
Using the Newton–Leibnitz formula.

$$\int_a^b f(x)\,dx = F(b) - F(a) \qquad \ldots \text{(i)}$$

Also, $$\int_b^a f(x)\,dx = F(a) - F(b) \qquad \ldots \text{(ii)}$$

From (i) and (ii), we get :

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

Hence proved.

PROPERTY - 3 :

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$$

Proof : Let $F(x)$ be indefinite integral of $f(x)$.
Using the Newton-Leibnitz formula,

$$\int_a^b f(t)\,dt = F(b) - F(a) \qquad \ldots \text{(i)}$$

Also, $$\int_a^b f(t)\,dt = F(b) - F(a) \qquad \ldots \text{(ii)}$$

From (i) and (ii), we get : $$\int_a^b f(x)\,dx = \int_a^b f(t)dt$$ Hence proved.

Illustration - 2

$$\int_{-1}^{2} |x|\,dx =$$

(A) $\frac{3}{2}$ (B) 2 (C) $\frac{5}{2}$ (D) 3

SOLUTION : (C)

$$\int_{-1}^{2} |x|\,dx = \int_{-1}^{0} |x|\,dx + \int_{0}^{2} |x|\,dx$$

[using property-1]

$$= \int_{-1}^{0} -x\,dx + \int_{0}^{2} x\,dx$$

[as $|x| = -x$ for $x < 0$ and $|x| = x$ for $x \geq 0$]

$$= -\left|\frac{x^2}{2}\right|_{-1}^{0} + \left|\frac{x^2}{2}\right|_{0}^{2}$$

$$= -\left(0 - \frac{1}{2}\right) + \left(\frac{4}{2} - 0\right) = \frac{5}{2}$$

Illustration - 3

$$\int_{-4}^{3} \left|x^2 - 4\right| dx =$$

(A) $\frac{7}{3}$ (B) $\frac{71}{3}$ (C) $\frac{80}{3}$ (D) $\frac{57}{3}$

SOLUTION : (B)

$$\int_{-4}^{3} |x^2 - 4|\,dx = \int_{-4}^{-2} |x^2 - 4|\,dx + \int_{-2}^{+2} |x^2 - 4|\,dx + \int_{2}^{3} |x^2 - 4|\,dx$$

$$= \int_{-4}^{-2} (x^2 - 4)\,dx + \int_{-2}^{2} (4 - x^2)\,dx + \int_{2}^{3} (x^2 - 4)\,dx$$

[as $|x^2 - 4| = 4 - x^2$ in $[-2, 2]$ and $x^2 - 4$ in other intervals]

$$= \left|\frac{x^3}{3} - 4x\right|_{-4}^{-2} + \left|4x - \frac{x^3}{3}\right|_{-2}^{2} + \left|\frac{x^3}{3} - 4x\right|_{2}^{3}$$

$$= \left(-\frac{8}{3} + 8\right) - \left(-\frac{64}{3} + 16\right) + \left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right) + \left(\frac{27}{3} - 12\right) - \left(\frac{8}{3} - 8\right) = \frac{71}{3}$$

2.2 Properties of Definite Integrals : (contd...)

PROPERTY - 4 :

$$\int_0^a f(x)dx = \int_0^a f(a-x)\,dx$$

Proof :

Let $I = \int_0^a f(x)\,dx$ substitute $x = a - t$ and $dx = -dt$

For $x = a,\ t = 0$ and for $x = 0,\ t = a$

$\Rightarrow \quad I = \int_a^0 f(a-t)(-dt) = \int_0^a f(a-t)\,dt$ [using property - 2]

We can replace by x using property - 3.

$\Rightarrow \quad I = \int_0^a f(a-x)\,dx$ Hence proved.

Illustration - 4 $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx =$

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{2}$ (C) π (D) 2π

SOLUTION : (A)

Let : $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$ . . . (i)

Using property - 4, we have :

$$I = \int_0^{\pi/2} \frac{\sqrt{\sin(\pi/2-x)}}{\sqrt{\sin(\pi/2-x)}+\sqrt{\cos(\pi/2-x)}}\,dx$$

$$I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx \quad . . . (ii)$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx + \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$$

$$\Rightarrow \quad 2I = \int_0^{\pi/2} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$$

$$\Rightarrow \quad 2I = \int_0^{\pi/2} dx = \frac{\pi}{2} \quad \Rightarrow \quad I = \frac{\pi}{4}$$

Illustration - 5

If $f(a-x) = f(x)$, then $\int_0^a x\, f(x)\, dx =$

(A) $a\int_0^a f(x)dx$ (B) $\frac{a}{2}\int_0^a f(x)dx$ (C) $a\int_0^{a/2} f(x)dx$ (D) $\frac{a}{2}\int_0^{a/2} f(x)dx$

SOLUTION : (B)

Let $I = \int_0^a x\, f(x)dx$

$\Rightarrow I = \int_0^a (a-x) f(a-x)\, dx$ [using property - 4]

$\Rightarrow I = \int_0^a (a-x) f(x)dx$ [using $f(x) = f(a-x)$]

$\Rightarrow \int_0^a af(x)dx - \int_0^a xf(x)dx$

$\Rightarrow I = a\int_0^a f(x)\, dx - I$

$\Rightarrow 2I = a\int_0^a f(x)\, dx$

$\Rightarrow I = \frac{a}{2}\int_0^a f(x)\, dx = \text{RHS}$

2.3 Properties of Definite Integrals : (contd......)

PROPERTY - 5 :

$$\int_0^{2a} f(x)\, dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$$

Proof :

Let $I = \int_0^{2a} f(x)\, dx$ $\Rightarrow I = \int_0^a f(x)dx + \int_a^{2a} f(x)\, dx$ [using property – 1]

Now in the second integral, put $x = 2a - t$

$\Rightarrow dx = -\, dt$

For $x = 2a$, $t = 0$ and for $x = a$, $t = a$

$\Rightarrow I = \int_0^a f(x)\, dx + \int_a^0 f(2a-t)(-dt)$ $\Rightarrow I = \int_0^a f(x)dx + \int_0^a f(2a-t)dt$ [using property – 2]

Replace t by x using property - 3, we get : $I=\int_0^a f(x)\,dx+\int_0^a f(2a-x)\,dx$ Hence proved

PROPERTY - 6 :

$\int_0^{2a} f(x)\,dx=0$ if $(2a-x)=-f(x)$; $\int_0^{2a} f(x)dx=2\int_0^a f(x)dx$ if $f(2a-x)=f(x)$

Proof : Consider Property - 5 i.e.

$$\int_0^{2a} f(x)\,dx=\int_0^a f(x)\,dx+\int_0^a f(2a-x)\,dx \quad \ldots\text{(i)}$$

If $f(2a-x)=-f(x)$, then (i) is reduced to : $\int_0^{2a} f(x)\,dx=\int_0^a f(x)\,dx-\int_0^a f(x)\,dx=0$

If $f(2a-x)=f(x)$, then (i) is reduced to :

$$\int_0^{2a} f(x)\,dx=\int_0^a f(x)\,dx+\int_0^a f(x)\,dx=2\int_0^a f(x)\,dx$$ Hence proved.

Illustration - 6

$$\int_0^{\pi}\frac{x}{1+\cos^2 x}\,dx=$$

(A) $\frac{\pi^2}{2}$ (B) $\frac{\pi^2}{\sqrt{2}}$ (C) $\frac{\pi^2}{2\sqrt{2}}$ (D) $\frac{\pi^2}{4}$

SOLUTION : (C)

Let $I=\int_0^{\pi}\frac{x}{1+\cos^2 x}\,dx \quad \ldots\text{(i)}$

$\Rightarrow I=\int_0^{\pi}\frac{(\pi-x)}{1+\cos^2(\pi-x)}dx$ [using property - 4]

$\ldots$ (ii)

Adding (i) and (ii), we get :

$$\Rightarrow 2I=\int_0^{\pi}\frac{\pi}{1+\cos^2 x}\,dx$$

$$\Rightarrow I=\frac{\pi}{2}\int_0^{\pi}\frac{dx}{1+\cos^2 x}=\frac{2\pi}{2}\int_0^{\pi/2}\frac{dx}{1+\cos^2 x}$$

[using property – 6]

Divide N^r and D^r by $\cos^2 x$ to get :

$$\Rightarrow \quad I = \pi \int_0^{\pi/2} \frac{\sec^2 x}{\sec^2 x + 1} dx$$

Put $\tan x = t \Rightarrow \sec^2 x \, dx = dt$

$[\sec^2 x = 1 + \tan^2 x]$

For $x = \pi/2,\ t \to \infty$ and for $x = 0,\ t = 0$

$$\Rightarrow \quad I = \pi \int_0^{\infty} \frac{dt}{2 + t^2}$$

$$\Rightarrow \quad I = \left| \frac{\pi}{\sqrt{2}} \tan^{-1} \frac{1}{\sqrt{2}} \right|_0^{\infty} = \frac{\pi}{\sqrt{2}} \times \frac{\pi}{2} = \frac{\pi^2}{2\sqrt{2}}$$

Illustration - 7

$$\int_0^{\pi/2} \log \sin x \, dx =$$

(A) $\frac{\pi}{2}\log(2)$ (B) $-\frac{\pi}{2}\log(2)$ (C) $\pi \log(2)$ (D) $-\pi \log(2)$

SOLUTION : (B)

Let $I = \int_0^{\pi/2} \log \sin x \, dx$ . . . (i)

$$\Rightarrow \quad I = \int_0^{\pi/2} \log \sin\left(\frac{\pi}{2} - x\right) dx \quad \text{[using property - 4]}$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \log \cos x \, dx \quad \text{. . . (ii)}$$

Adding (i) and (ii) we get :

$$2I = \int_0^{\pi/2} \log(\sin x \cos x) \, dx = \int_0^{\pi/2} \log\left(\frac{\sin 2x}{2}\right) dx$$

$$\Rightarrow \quad 2I = \int_0^{\pi/2} \log \sin 2x \, dx - \int_0^{\pi/2} \log 2 \, dx$$

$$\Rightarrow \quad 2I = \int_0^{\pi/2} \log \sin 2x \, dx - \frac{\pi}{2} \log 2 \quad \text{. . . (iii)}$$

Let $I_1 = \int_0^{\pi/2} \log \sin 2x \, dx$

Put $t = 2x \Rightarrow dt = 2dx$

For $x = \frac{\pi}{2},\ t = \pi$ and for $x = 0,\ t = 0$

$$\Rightarrow \quad I_1 = \frac{1}{2} \int_0^{\pi} \log \sin t \, dt = \frac{2}{2} \int_0^{\pi/2} \log \sin t \, dt$$

$$\Rightarrow \quad I_1 = \int_0^{\pi/2} \log \sin x \, dx \quad \text{[using property - 3]}$$

$\Rightarrow \quad I_1 = I$

Substituting in (iii) we get :

$2I = I - \pi/2 \log 2$

$\Rightarrow \quad I = -\pi/2 \log 2$

[learn this result so that you can directly apply it in other difficult problem]

Illustration - 8 *Which of the following statements is(are) true ?*

(A) $\int_0^{\pi/2} f(\sin 2x)\sin x\,dx = \int_0^{\pi/2} f(\sin 2x)\cos x\,dx$ (B) $\int_0^{\pi/2} f(\sin 2x)\sin x\,dx = \int_0^{\pi/4} f(\sin 2x)\cos x\,dx$

(C) $\int_0^{\pi/2} f(\sin 2x)\sin x\,dx = \int_0^{\pi/4} f(\cos 2x)\cos x\,dx$ (D) $\int_0^{\pi/2} f(\sin 2x)\sin x\,dx = \sqrt{2}\int_0^{\pi/4} f(\cos 2x)\cos x\,dx$

SOLUTION : (AD)

Let $I = \int_0^{\pi/2} f(\sin 2x)\sin x dx$ . . . (i)

$\Rightarrow I = \int_0^{\pi/2} f\left[\sin 2(\pi/2 - x)\right]\sin(\pi/2 - x)\,dx$

[using property - 4]

$\Rightarrow I = \int_0^{\pi/2} f\left[\sin(\pi - 2x)\right]\cos x\,dx$

$\Rightarrow I = \int_0^{\pi/2} f(\sin 2x)\cos x\,dx$ . . . (ii)

$I = \int_0^{\pi/2} f(\sin 2x)\sin x dx$

$= \int_0^{\pi/4} f(\sin 2x)\sin x\,dx + \int_0^{\pi/4} f\left[\sin 2(\pi/2 - x)\right]$

$\sin(\pi/2 - x)\,.\,dx$ [using property - 5]

$= \int_0^{\pi/4} f(\sin 2x)\sin x\,dx + \int_0^{\pi/4} f(\sin 2x)\cos x\,dx$

$= \int_0^{\pi/4} f(\sin 2x)(\sin x + \cos x)\,dx$

$= \int_0^{\pi/4} f\left[\sin 2(\pi/4 - x)\right]$

$[\sin(\pi a/4 - x) + \cos(\pi/4 - x)dx$

[using property - 6]

$= \int_0^{\pi/4} f(\cos 2x)$

$\left[\frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x + \frac{1}{\sqrt{2}}\sin x\right]dx$

$= \sqrt{2}\int_0^{\pi/4} f(\cos 2x)\cos x\,dx$

2.4 Properties of Definite Integrals : (contd ......)

PROPERTY 7 :

$$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$$

Proof :

Let $I = \int_a^b f(a+b-x)dx$

Put $a+b-x=t \Rightarrow dx=-dt$ when $x = a,\ t = b$ and When $x = b,\ t = a$

$\Rightarrow \quad I = \int_b^a f(t)\,(-dt) = \int_a^b f(t)\,dt$ [using property – 2]

$\Rightarrow \quad I = \int_a^b f(x)\,dx$ Hence proved.

PPOPERTY – 8 :

$$\int_{-a}^{a} f(x)dx = \int_0^a [f(x)+f(-x)]dx$$

$$= 2\int_0^a f(x)dx \quad \text{if} \quad f(x) \text{ is even} \quad \text{i.e.} \quad f(-x)=f(x)$$

$$= 0 \quad \text{if} \quad f(x) \text{ is odd} \quad \text{i.e.} \quad f(-x)=-f(x)$$

Proof :

Consider property – 1 , i.e.

$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_0^a f(x)dx \quad \ldots \text{(i)}$$

Consider $I = \int_{-a}^{0} f(x)\,dx$

Put $x=-t \Rightarrow dx=-dt.$ When $x=-a,\ t=a$ and When $x=0,\ t=0.$

$$\Rightarrow \quad I = \int_a^0 f(-t)(-dt) = \int_0^a f(-t)dt = \int_0^a f(-x)dx \quad \ldots \text{(ii)}$$ [using properties – 2 and 3]

Combining (i) and (ii), we get :

$$\int_{-a}^{a} f(x)dx = \int_0^a [f(x)+f(-x)]dx$$

If $f(x)$ is even function, then $f(-x) = f(x)$ $\Rightarrow$ $\int_{-a}^{a} f(x)dx = \int_0^a [f(x) + f(x)]dx = 2\int_0^a f(x)dx$

$\Rightarrow$ $\int_{-a}^{a} f(x)dx = \int_0^a [f(x) - f(x)]dx = 0$ Hence proved.

PROPERTY – 9 :

$\int_0^{nT} f(x)dx = n\int_0^T f(x)dx$ where $f(x)$ is a periodic function with period T and n is an integer.

Let $I = \int_0^{nT} f(x)dx$

$$= \int_0^T f(x)dx + \int_T^{2T} f(x)dx + \ldots + \int_{rT}^{(r+1)T} f(x)dx + \ldots.. + \int_{(n-1)T}^{nT} f(x)dx$$

Consider any one of the integrals of RHS.
In general, let us take

$$I_r = \int_{rT}^{(r+1)T} f(x)dx \qquad \text{(where } 0 \le r \le n-1\text{)}$$

Put $x = rT + y$ $\Rightarrow$ $dx = dy$

For $x = rT, y = 0$ and *for* $x = (x+1)T,\ y = T.$

$$I_r = \int_0^T f(rT + y)dy = \int_0^T f(y)dy = \int_0^T f(x)dx$$

Hence all integrals in RHS are equal to $\int_0^T f(x)dx.$

$\Rightarrow$ $I = \int_0^T f(x)dx + \int_0^T f(x)dx + \ldots\ldots. + n$ times

$\Rightarrow$ $I = n\int_0^T f(x)dx$ Hence proved.

Illustration - 9

$$\int_0^{\pi} \frac{x \sin(2x) \sin\left(\frac{\pi}{2}\cos x\right) dx}{2x-\pi} =$$

(A) $\frac{2}{\pi^2}$ (B) $\frac{4}{\pi^2}$ (C) $\frac{8}{\pi^2}$ (D) $\frac{8}{\pi}$

SOLUTION : (C)

Let $I = \int_0^{\pi} \frac{x \sin(2x) \sin\left(\frac{\pi}{2}\cos x\right) dx}{2x-\pi}$ . . . (i)

Apply property - 4 to get

$\Rightarrow$

$$I = \int_0^{\pi} + \frac{(\pi - x)\sin(2\pi - 2x)\sin\left(\frac{\pi}{2}\cos(\pi - x)\right) dx}{2(\pi - x) - \pi}$$

$$= \int_0^{\pi} - \frac{(\pi - x)\sin 2x \sin\left(\frac{\pi}{2}\cos x\right) dx}{2x - \pi}$$ . . . (ii)

Add (i) and (ii) to get

$$2I = \int_0^{\pi} \sin 2x \sin\left[\frac{\pi}{2}\cos x\right] dx$$

Let $\frac{\pi}{2}\cos x = t \Rightarrow -\frac{\pi}{2}\sin x\, dx = dt$

$$\Rightarrow I = \frac{4}{\pi^2}\int_{-\pi/2}^{\pi/2} t \sin t\, dt = \frac{8}{\pi^2}\int_0^{\pi/2} t \sin t\, dt$$

$$\Rightarrow I = \frac{8}{\pi^2}\left[t\int_0^{\pi/2} \sin t\, dt + \int_0^{\pi/2} \cos t\, dt\right]$$

$$\Rightarrow I = \frac{8}{\pi^2}\left[-\left|t \cos t\right|_0^{\pi/2} + (\sin t)_0^{\pi/2}\right]$$

$$= \frac{8}{\pi^2}[0+1] = \frac{8}{\pi^2}$$

Illustration - 10

If $\int_2^3 x\sqrt{5-x}\, dx = \frac{10}{3}(3\sqrt{3} - 2\sqrt{2}) - \frac{2}{5}(9\sqrt{3} - k)$, *then* $k =$

(A) $2\sqrt{2}$ (B) $4\sqrt{2}$ (C) $\sqrt{2}$ (D) $3\sqrt{2}$

SOLUTION : (B)

Let $I = \int_2^3 x\sqrt{5-x}\, dx$

$$\Rightarrow I = \int_2^3 (2+3-x)\sqrt{5-(2+3-x)}\, dx$$

[using property – 7]

$$\Rightarrow I = \int_2^3 (5-x)\sqrt{x}\, dx$$

$$\Rightarrow I = \int_2^3 5\sqrt{x}\, dx - \int_2^3 x\sqrt{x}\, dx$$

$$\Rightarrow I = 5\left|\frac{2}{3}x\sqrt{x}\right|_2^3 - \frac{2}{5}\left|x^2\sqrt{x}\right|_2^3$$

$$\Rightarrow I = \frac{10}{3}(3\sqrt{3} - 2\sqrt{2}) - \frac{2}{5}(9\sqrt{3} - 4\sqrt{2})$$

Illustration - 11 $\int_a^b \frac{f(x)}{f(x)+f(a+b-x)}dx$

(A) $\frac{a+b}{2}$ (B) $\frac{a-b}{2}$ (C) $\frac{b-a}{2}$ (D) $b-a$

SOLUTION : (C)

Let $I=\int_a^b \frac{f(x)}{f(x)+f(a+b-x)}\,dx$ . . . (i)

$\Rightarrow I=\int_a^b \frac{f(a+b-x)}{f(a+b-x)+f[a+b-(a+b-x)]\,dx}$

$\Rightarrow I=\int_a^b \frac{f(a+b-x)}{f(a+b-x)+f(x)}\,dx$ . . . (ii)

Adding (i) and (ii), we get

$\Rightarrow 2I=\int_a^b \frac{f(x)+f(a+b-x)}{f(x)+f(a+b-x)}\,dx$

$\Rightarrow 2I=\int_a^b dx=b-a \quad \Rightarrow \quad I=\frac{b-a}{2}$

Illustration - 12 $\int_{-1}^{+1} log\left(\frac{2-x}{2+x}\right) sin^2 x\,dx =$

(A) 0 (B) log 2 (C) 1 (D) 2 log 2

SOLUTION : (A)

Let $f(x)=\log\left(\frac{2-x}{2+x}\right)\sin^2 x\,dx$

$\Rightarrow f(-x)=\log\left(\frac{2+x}{2-x}\right)\sin^2(-x)$

$\Rightarrow f(-x)=\log\left[\left(\frac{2-x}{2+x}\right)^{-1}\right]\sin^2 x$

$=-\log\left(\frac{2-x}{2+x}\right)\sin^2 x=-f(x)$

$\Rightarrow f(x)$ is an odd function.

$\therefore \int_{-1}^{1} \log\left(\frac{2-x}{2+x}\right)\sin^2 dx=0$

Illustration - 13

The value of $\int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x}\,dx$ *is :*

(A) π^2 **(B)** $2\pi^2$ **(C)** $4\pi^2$ **(D)** 0

SOLUTION : (A)

$$I = \int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x}\,dx = 2\int_0^{\pi} \frac{2x\sin x}{1+\cos^2 x}.dx$$

$$\left[\text{using : } \int_{-a}^{a} f(x)dx = \int_0^a [f(x)+f(-x)]dx\right]$$

$$\Rightarrow \quad I = 4\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx$$

$$\Rightarrow \quad 2I = 4\int_0^{\pi} \frac{\pi\sin x}{1+\cos^2 x}\,dx$$

$$\left[\text{using : } \int_0^a f(x)dx = \int_0^a f(a-x)\,dx\right]$$

$$\Rightarrow \quad I = 4\pi\int_0^{\pi/2} \frac{\sin x\,dx}{1+\cos^2 x}$$

$$\left[\text{using : } \int_0^{2a} f(x)dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx\right]$$

Put $\cos x = t$ $\Rightarrow$ $-\sin x\,dx = dt$

For $x = 0,\ t = 1$ and for $x = \pi/2,\ t = 0$

$$\Rightarrow \quad I = 4\pi\int_0^1 \frac{dt}{1+t^2} = 4\pi\tan^{-1}t\Big|_0^1 = 4\pi\frac{\pi}{4} = \pi^2$$

Illustration - 14

If $\int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4}\cos^{-1}\frac{2x}{1+x^2}\,dx = -\frac{\pi}{\sqrt{3}} - \frac{\pi}{4}\log\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right) + k$, *then* $k =$

(A) $\frac{\pi^2}{6}$ **(B)** $\frac{\pi^2}{8}$ **(C)** $\frac{\pi^2}{12}$ **(D)** $\frac{\pi^2}{2}$

SOLUTION : (C)

$$I = \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4}\cos^{-1}\frac{2x}{1+x^2}\,dx$$

$$= \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4}\left[\frac{\pi}{2} - \sin^{-1}\frac{2x}{1+x^2}\right]dx$$

[using : $\sin^{-1}x + \cos^{-1}x = \pi/2$]

$$\Rightarrow \quad I = \frac{\pi}{2}\int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4}\,dx - \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4}\sin^{-1}\frac{2x}{1+x^2}\,dx$$

As integrand of second integral is an odd function, integral will be zero i.e.

$$\Rightarrow \quad I = \frac{\pi}{2}\int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4}\,dx - 0$$ [using property - 8]

$$= -\frac{2\pi}{2}\int_0^{1/\sqrt{3}} \frac{x^4-1+1}{x^4-1}\,dx = -\pi\int_0^{1/\sqrt{3}}\left(1+\frac{1}{x^4-1}\right)dx$$

$$\Rightarrow \quad I = \frac{-\pi}{\sqrt{3}} + \frac{(-\pi)}{2}\int_0^{1/\sqrt{3}} \frac{x^2+1-(x^2-1)}{(x^2+1)\,(x^2-1)}\,dx$$

$$= -\frac{-\pi}{\sqrt{3}} - \frac{\pi}{2}\left[\int_0^{1/\sqrt{3}} \frac{1}{x^2-1} - \int_0^{1/\sqrt{3}} \frac{1}{x^2+1}\,dx\right]$$

$$= -\frac{-\pi}{\sqrt{3}} - \frac{\pi}{2}\left[\frac{1}{2}\left|\log - \left(\frac{x-1}{x+1}\right)\right|_0^{1/\sqrt{3}} - \left|\tan^{-1}x\right|_0^{1/\sqrt{3}}\right]$$

$$= -\frac{\pi}{\sqrt{3}} + \frac{\pi^2}{12} - \frac{\pi}{4}\log\frac{\sqrt{3}-1}{\sqrt{3}+1}$$

Illustration - 15

$$\int_0^{\pi/2} \sqrt{1-\sin 2x}\,dx =$$

(A) 0 **(B)** $2\sqrt{2}-1$ **(C)** $\sqrt{2}-1$ **(D)** $2\sqrt{2}-2$

SOLUTION : (D)

Let $I = \int_0^{\pi/2} \sqrt{1-\sin 2x}\,dx$

$$\Rightarrow \quad I = \int_0^{\pi/2} \sqrt{(\sin x - \cos x)^2}\,dx$$

$$\Rightarrow \quad I = \int_0^{\pi/2} |\sin x - \cos x|\,dx$$

$$\Rightarrow \quad I = \int_0^{\pi/4} |\sin x - \cos x|\,dx + \int_{\pi/4}^{\pi/2} |\sin x - \cos x|\,dx$$

$$\Rightarrow \quad I = \int_0^{\pi/4} (\cos x - \sin x)\,dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)\,dx$$

$$\Rightarrow \quad I = |\sin x + \cos x|_0^{\pi/4} + |-\cos x - \sin x|_{\pi/4}^{\pi/2}$$

$$\Rightarrow \quad I = \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 1\right) + (-1) - \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow \quad I = 2\sqrt{2} - 2.$$

Illustration - 16

Given a function such that :

(i) *it is integrable over every interval on the real line.*

(ii) $f(t+x) = f(x)$ *for every x and a real t, then the integral* $\int_a^{a+t} f(x)\,dx$ *is independent on :*

(A) *a only* **(B)** *t only* **(C)** *Both a and t* **(D)** *Neither of a and t*

SOLUTION : (B)

Let $I = \int_a^{a+t} f(x)\,dx$

$= \int_a^t f(x)\,dx + \int_t^{a+t} f(x)\,dx$ . . . (i)

Consider $I_1 = \int_t^{a+t} f(x)\,dx$

Put $x = y + t \Rightarrow dx = dy$

For $x = a + t, y = a$ and For $x = t, y = 0$.

$\Rightarrow I_1 = \int_0^a f(y+t)dy$

$\Rightarrow I_1 = \int_0^a f(y)\,dy$ [using $f(x+T) = f(x)$]

$\Rightarrow I_1 = \int_0^a f(x)\,dx$ [using property 3]

On substituting the value of I_1 (i), we get :

$I = \int_a^t f(x)\,dx + I_1$

$\Rightarrow I = \int_a^t f(x)\,dx + \int_0^a f(x)\,dx$

$\Rightarrow I = \int_0^t f(x)\,dx$ [using property-1]

$\Rightarrow$ I is independent of a.

Illustration - 17

If $\int_0^{n\pi+v} |\sin x|\,dx = k - \cos v$, *where n is a +ve integer and* $0 \le v \le \pi$, *then* $k =$

(A) n **(B)** $n + 1$ **(C)** $2n$ **(D)** $2n + 1$

SOLUTION : (D)

Let

$I = \int_0^{n\pi+v} |\sin x|\,dx = \int_0^{n\pi} |\sin x|\,dx + \int_{n\pi}^{n\pi+v} |\sin x|\,dx$

[using property - 1]

$\Rightarrow I = I_1 + I_2$ . . . (i)

Consider I_1:

$I_1 = \int_0^{n\pi} |\sin x|\,dx = n\int_0^{\pi} |\sin x|\,dx$

[using property - 9 and period of $|\sin x|$ is π]

$\Rightarrow I_1 = n\int_0^{\pi} \sin x\,dx$

[As $\sin x \ge 0$ in $[0, \pi]$, $|\sin x| = \sin x$]

$\Rightarrow I_1 = -n\,|\cos x|_0^{\pi} = -n[-1-1] = 2n$

Consider I_2: $I_2 = \int_{n\pi}^{n\pi+v} |\sin x|\,dx$

Put $x = n\pi + \theta \Rightarrow dx = d\theta$

When x is $n\pi$, $\theta = 0$ and when $x = n\pi + v$, $\theta = +v$.

$\Rightarrow I_2 = \int_0^v |\sin(n\pi+\theta)|\,d\theta = \int_0^v |\sin\theta|\,d\theta$

[as period of $|\sin x| = \pi$]

$\Rightarrow I_2 = \int_0^v |\sin\theta|\,d\theta = \int_0^v \sin\theta\,d\theta$

[as for $0 \le \theta \le \pi$, $\sin\theta$ is positive]

$= -|\cos\theta|_0^v = 1 - \cos v$

On substituting the values of I_1 and I_2 in (i), we get

$I = 2n + (1 - \cos v) = 2n + 1 - \cos v$.

Illustration - 18 *It is known that $f(x)$ is an odd function in the interval $\left[\frac{-T}{2}, \frac{T}{2}\right]$ and has a period equal to T. $\int_a^x f(t)\,dt$ is also periodic function with the period =*

(A) $\frac{T}{2}$ (B) T (C) $2T$ (D) *None of these*

SOLUTION : (B)

It is given that : $f(-x) = -f(x)$ . . . (i)

and $f(x+T) = f(x)$ . . . (ii)

Let $g(x) = \int_a^x f(t)\,dt.$

$\Rightarrow \quad g(x+T) = \int_a^{x+T} f(t)\,dt$

$$= \int_a^x f(t)\,dt + \int_x^{T/2} f(t)\,dt + \int_{T/2}^{x+T} f(t)\,dt$$

[using properly - 1]

Put $t = y + T$ in the third integral on RHS.

$\Rightarrow dt = dy$

when $t = T/2$, $y = -T/2$ and when $t = x + T$, $y = x$

$$\Rightarrow \quad g(x+T) = \int_a^x f(t)\,dt + \int_x^{T/2} f(t)\,dt + \int_{-T/2}^{x} f(y+T)\,dy$$

Using (ii), we get

$$g(x+T) = \int_a^x f(t)\,dt + \int_x^{T/2} f(t)\,dt + \int_{-T/2}^{x} f(y)\,dy$$

$$g(x+T) = \int_a^x f(t)\,dt + \int_{-T/2}^{T/2} f(t)\,dt$$

[using properly - 1]

$$\Rightarrow \quad g(x+T) = \int_a^x f(t)\,dt$$

[using properly - 8]

$\Rightarrow \quad g(x+T) = g(x)$

$\Rightarrow \quad g(x)$ is also a periodic function with period T.

Illustration - 19 *A positive integer $n \le 5$ such that : $\int_0^1 e^x (x-1)^n dx = 16 - 6e$ is :*

(A) 2 (B) 3 (C) 4 (D) 5

SOLUTION : (B)

Let $I_n = \int_0^1 e^x (x-1)^n\,dx$

$$I_n = \left[(x-1)^n \int e^x dx\right]_0^1 - \int_0^1 e^x n\,(x-1)^{n-1}\,dx$$

[using integration by parts]

$\Rightarrow \quad I_n = 0 - (-1)^n - n\int_0^1 e^x (x-1)^{n-1} dx$

$\Rightarrow \quad I_n = -(-1)^n - n\, I_{n-1} \qquad \ldots \text{(i)}$

Also $\quad I_0 = \int_0^1 e^x (x+1)^0 dx = e - 1$

$\Rightarrow \quad I_1 = 1 - I_0 = 1 - (e-1) = 2 - e \qquad$ [using (i)]

$\Rightarrow \quad I_2 = -1 - 2I_1 = -1 - 2(2-e) = -5 + 2e$

$\Rightarrow \quad I_3 = 1 - 3I_2 = 1 - 3(-5+2e) = 16 - 6e$

$\Rightarrow$ Hence for $n = 3$ $\int_0^1 e^x (x-1)^m dx = 16 - 6e$

2.5. Properties of Definite Integrals : (contd....)

PROPERTY - 10 :

$$\int_0^\infty f(x)\,dx = \lim_{n\to\infty}\int_0^n f(x)\,dx \quad ; \quad \int_{-\infty}^a f(x)\,dx = \lim_{n\to-\infty}\int_n^a f(x)\,dx$$

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^0 f(x)\,dx + \int_0^\infty f(x)\,dx = \lim_{a\to-\infty}\int_a^0 f(x)\,dx + \lim_{b\to\infty}\int_0^b f(x)\,dx$$

The integral $\int_{-\infty}^{\infty} f(x)\,dx$ converges if both of the above integrals converges.

PROPERTY - 11 : $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$.

PROPERTY - 12 :

If the function $f(x)$ and $g(x)$ are defined on $[a, b]$ and differentiable at all points

$x \in [a, b]$, then $\dfrac{d}{dx}\left[\int_{f(x)}^{g(x)} h(t)\,dt\right] = h[g(x)]\,g'(x) - h[f(x)]\,f'(x)$

PROPERTY - 13 :

If $f(x) \ge g(x)$ for all $x \in [a, b]$, then $\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx.$

Put $g(x) = 0$ for all $x \in [a, b]$ in above property to get another useful property, i.e.

If $f(x) \ge 0$ for all $x \in [a, b]$, then $\int_a^b f(x)dx \ge 0$.

Illustration - 20

If $f(x) = \int_{x^2}^{x^3} \frac{1}{\log t}\, dt \quad t > 0,$ *then* $f'(x) =$

(A) $\frac{x^3 - x^2}{\log x}$ (B) $\frac{x^2 - x}{\log x}$ (C) $\frac{x^3 - x}{\log x}$ (D) *None of these*

SOLUTION : (B)

Using the property - 12,

$$f'(x) = \frac{1}{\log(x^3)} \frac{d}{dx}(x^3) - \frac{1}{\log x^2} \frac{d}{dx}(x^2)$$

$$\Rightarrow \quad f'(x) = \frac{3x^2}{3\log x} - \frac{2x}{2\log x} = \frac{x^2 - x}{\log x}$$

Illustration - 21 *The total number of points of local minimum and local maximum of the function* $f(x) = \int_0^{x^2} \frac{t^2 - 5t + 4}{2 + e^t}\, dt$ *is :*

(A) 2 (B) 3 (C) 4 (D) 5

SOLUTION : (D)

Let $y = \int_0^{x^2} \frac{t^2 - 5t + 4}{2 + e^t}\, dt = \int_0^{x^2} \frac{(t-1)(t-4)}{2 + e^t}\, dt$

For the points of Extremes,

$$\frac{dy}{dx} = 0 \Rightarrow \left[\frac{(x^2 - 1)(x^2 - 4)}{2 + e^{x^2}}\right](2x) = 0 \qquad \text{[using property-12]}$$

$\Rightarrow \quad x = 0 \quad$ or $\quad x^4 - 5x^2 + 4 = 0$

$\Rightarrow \quad x = 0 \quad$ or $\quad (x-1)(x+1)(x-2)(x+2) = 0$

$\Rightarrow \quad x = 0, \quad x = \pm 1$ and $x = \pm 2$

With the help of first derivative test, check your self that $x = -2, 0, 2$ are points of local minimum and $x = -1, 1$ are points of local maximum.

IMPORTANT RESULTS Section - 3

3.1 Definite Integral as a limit of a Sum

We can express definite integral as a limit of the sum of a certain number of terms. Let $f(x)$ be a continuous function in the interval $[a, b]$. Divide a-b interval into n equal parts such that width of each part is h.

$\Rightarrow \quad nh = b - a.$

The definite integral of a function $f(x)$ in the interval $[a, b]$ can be defined as :

$$\int_a^b f(x)\,dx = \lim_{\substack{n\to\infty \\ h\to 0}} h\left[f(a+h) + f(a+2h) + \ldots\ldots + f(a+nh)\right] \qquad \text{where } nh = b - a.$$

$$= \lim_{\substack{n\to\infty \\ h\to 0}} h\sum_{r=1}^{n} f(a+rh) \qquad \text{where } nh = b - a.$$

With the help of this formula, we can evaluate some simple definite Integrals. The process of finding definite integrals with the use of above formula is known as *definite Integral as a limit of a sum or definite Integral by first principle*.

3.2 Summation of Series with help of definite integrals

Consider the "limit of a sum" formula defined in the previous section ie.

$$\int_a^b f(x)\,dx = \lim_{\substack{n\to\infty \\ h\to 0}} h\sum_{r=1}^{n} f(a+rh) \qquad \text{where } nh = b - a. \qquad \ldots\text{(i)}$$

Put $a = 0$ and $b = 1$, $\Rightarrow \quad nh = 1 \quad \Rightarrow \quad h = 1/n.$

Put $a = 0$ and $b = 1$ and $h = 1/n$ in (i) to get :

$$\int_0^1 f(x)\,dx = \lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n} f\left(\frac{r}{n}\right) \qquad \ldots\text{(ii)}$$

With the help of formula - 2, we can evaluate the sum to infinite terms of certain series.

Working Rule :

First of all express the given series in the form

$\frac{1}{n}\sum f\left(\frac{r}{n}\right)$ and then replace the integral sign $\int$ for $\sum$ and $\frac{r}{n}$ by x.

The lower and upper limit of integration are the values of $\lim_{n\to\infty}\left(\frac{r}{n}\right)$ for the least and the greatest values of r respectively.

3.3 Estimation of a Definite Integral

If $f(x)$ is a function defined in the interval $[a, b]$ then :

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$

where m is the least and M is the greatest value of the function $f(x)$ in the interval $[a, b]$.

3.4 Mean value theorem of definite Integrals

If the function $f(x)$ is continuous in the interval $[a, b]$, then :

$$\int_a^b f(x)\,dx = f(c)(b-a), \qquad \text{where } a < c < b.$$

3.5 Two useful Formulae

1. If n be a positive integer, then :

$$\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx$$

$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}, \qquad \text{when } n \text{ is even}$$

$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\cdot\frac{4}{5}\cdot\frac{2}{3}\cdot 1, \qquad \text{when } n \text{ is odd}$$

2. $$\int_0^{\pi/2} \sin^m x \cos^n x\,dx = \int_0^{\pi/2} \sin^n x \cos^m x\,dx$$

$$= \frac{(m-1).(m-3)\ldots\ldots(1\text{ or }2)\,(n-1).(n-3)\ldots\ldots(1\text{ or }2)}{(m+n).(m+n-2)\ldots\ldots(1\text{ or }2)}\,\frac{\pi}{2},$$

when both m and n $\in$ even integer

$$= \frac{(m-1).(m-3)\ldots\ldots(1\text{ or }2)\,(n-1).(n-3)\ldots\ldots(1\text{ or }2)}{(m+n).(m+n-2)\ldots\ldots(1\text{ or }2)},$$

otherwise

Ilustrating the Concepts :

Evaluate : $\int_a^b x^2\, dx$ *using limit of a sum formula.*

Let $I = \int_a^b x^2 dx = \lim_{\substack{n \to \infty \\ h \to 0}} h\,[(a+h)^2 + (a+2h)^2 + \ldots\ldots\ldots\ldots + (a+nh)^2]$

$$\Rightarrow \quad I = \lim_{\substack{n \to \infty \\ h \to 0}} \left[nha^2 + \frac{2\,ah^2 n\,(n+1)}{2} + \frac{h^3 n\,(n+1)\,(2n+1)}{6} \right]$$

Using $nh = b - a$, we get :

$$\Rightarrow \quad I = \lim_{n \to \infty} \left[a^2 (b-a) + a\,(b-a)^2 \left(1 + \frac{1}{n}\right) + (b-a)^3 \frac{1}{6} \left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) \right]$$

$$\Rightarrow \quad I = a^2 (b-a) + a\,(b-a)^2 + \frac{(b-a)^2}{6}(2)$$

$$\Rightarrow \quad I = (b-a)\left[a^2 + ab - a^2 + \frac{b^2 + a^2 - 2ab}{3} \right]$$

$$\Rightarrow \quad I = \frac{(b-a)}{3}\left[a^2 + b^2 + ab \right] = \frac{b^3 - a^3}{3}$$

Illustration - 22 *The sum* $S = \lim_{n \to \infty} \left[\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3n} + \ldots\ldots + \frac{1}{2n} \right]$ *is equal to :*

(A) $\log 2$ **(B)** $\log 3$ **(C)** $\frac{\log 2}{2}$ **(D)** $\frac{\log 3}{2}$

SOLUTION : (A)

$$\Rightarrow \quad S = \lim_{n \to \infty} \left[\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \ldots + \frac{1}{n+n} \right]$$

$$\Rightarrow \quad S = \lim_{n \to \infty} \frac{1}{n} \left[\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{n}{2n} \right]$$

$$\Rightarrow \quad S = \lim_{n \to \infty} \frac{1}{n} \left[\frac{1}{1+1/n} + \frac{1}{1+2/n} + \ldots\ldots + \frac{1}{1+n/n} \right]$$

$$\Rightarrow S = \lim_{n \to \infty} \frac{1}{n} \left[\sum_{r=1}^{n} \frac{1}{1+r/n} \right] \quad \Rightarrow \quad S = \int_0^1 \frac{1}{1+x}\, dx$$

$$\Rightarrow \quad S = \left| \log\,(1+x) \right|_0^1 = \log 2.$$

Illustration - 23 *The sum of the series :* $\lim_{n\to\infty}\left(\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\ldots\ldots\ldots+\frac{1}{6n}\right)$ *is equal to :*

(A) $\ell n\ 5$ **(B)** $\ell n\ 6$ **(C)** $\frac{\ell n\ 5}{2}$ **(D)** $\frac{\ell n\ 6}{2}$

SOLUTION : (B)

Let $S=\lim_{n\to\infty}\left(\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\ldots\ldots+\frac{1}{6n}\right)$

Take $1/n$ common from the series i.e.

$$S=\lim_{n\to\infty}\frac{1}{n}\left[\frac{1}{1+1/n}+\frac{1}{1+2/n}+\ldots\ldots+\frac{1}{1+5n/n}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{5n}\frac{1}{1+r/n}$$

For the definite integral,

$$\text{Lower limit }=a=\lim_{n\to\infty}\left(\frac{r}{n}\right)=\lim_{n\to\infty}\frac{1}{n}=0$$

$$\text{Upper limit }=b=\lim_{n\to\infty}\left(\frac{r}{n}\right)=\lim_{n\to\infty}\frac{5n}{n}=5$$

Therefore, $S+\lim_{n\to\infty}\sum_{r=0}^{5n}\frac{1}{1+(r/n)}$

$$=\int_0^5\frac{dx}{1+x}=\ln\left|1+x\right|_0^5=\ln 6-\ln 1=\ln 6$$

Illustration - 24 *Which of the following is true ?*

(A) $\int_0^1 e^{x^2}\,dx\le 1$ **(B)** $\int e^{x^2}\,dx\ge e$ **(C)** $1\le\int_0^1 e^{x^2}\,dx\le e$ **(D)** *None of these*

SOLUTION : (C)

Using the result given in Section 3.3,

$$m\,(1-0)\le\int_0^1 e^{x^2}\,dx\le M\,(1-0)\qquad\ldots\text{(i)}$$

let $f(x)=e^{x^2}$

$\Rightarrow f'(x)=2xe^{x^2}=0\quad\Rightarrow\quad x=0.$

Apply first derivative test to check that there exists a local minimum at $x=0$.

$\Rightarrow$ $f(x)$ is an increasing function in the interval $[0, 1]$

$\Rightarrow$ $m=f(0)=1$ and $M=f(1)=e^1=e$

Substituting the values of m and M in (i), we get

$$(1-0)\le\int_0^1 e^{x^2}\,dx\le M\,(1-0)$$

$$\Rightarrow\quad 1\le\int_0^1 e^{x^2}\,dx\le e$$

Ilustrating the Concepts :

(i) *Consider the integral :* $I = \int_0^{2\pi} \frac{dx}{5-2\cos x}$. *Making the substitution* $\tan x/2 = t$, *we have :*

$$\int_0^{2\pi} \frac{dx}{5-2\cos x} = \int_0^0 \frac{2\,dt}{(1+t^2)\left[5-2\frac{1-t^2}{1+t^2}\right]} = 0$$

This result is obviously wrong since the integrand is positive and consequently the integral of this function can not be equal to zero. Find the mistake in this evaluation.

The mistake lies in the substitution $\tan\frac{x}{2} = t$. *Since the function* $\tan\frac{x}{2}$ *is discontinuous at* $x = \pi$, *a point in the interval* $(0, 2\pi)$, *we can not use this substitution for the changing the variable of integration.*

(ii) *Find the mistake in the following evaluation of the integral.*

$$\int_0^{\pi} \frac{dx}{1+2\sin^2 x} = \int_0^{\pi} \frac{dx}{\cos^2 x+3\sin^2 x} = \int_0^{\pi} \frac{\sec^2 x\,dx}{1+3\tan^2 x} = \frac{1}{\sqrt{3}}\left[\tan^{-1}\left(\sqrt{3}\tan x\right)\right]_0^{\pi} = 0$$

The Newton-Leibnitz formula for evaluating the definite integrals is not applicable here since the anti-derivative, $F(x) = \frac{1}{\sqrt{3}}\tan^{-1}\left(\sqrt{3}\tan x\right)$ has a discontinuity at the point $x = \pi/2$ which lies in the interval $[0, \pi]$.

$$\underset{at\ x=\pi/2}{\text{LHL}} = \lim_{h\to 0}\frac{1}{\sqrt{3}}\tan^{-1}\sqrt{3}\left[\tan\left(\frac{\pi}{2}-h\right)\right] = \lim_{h\to 0}\frac{1}{\sqrt{3}}\tan^{-1}\left(\sqrt{3}\cot h\right)$$

$$= \lim_{h\to 0}\frac{1}{\sqrt{3}}\tan^{-1}(\to\infty) = \frac{\pi}{2\sqrt{3}} \qquad \textbf{. . . (i)}$$

$$\underset{at\ x=\pi/2}{\text{RHL}} = \lim_{h\to 0}\frac{1}{\sqrt{3}}\tan^{-1}\left[\tan\left(\frac{\pi}{2}+h\right)\right] = \lim_{h\to 0}\frac{1}{\sqrt{3}}\tan^{-1}\left(-\sqrt{3}\cot h\right)$$

$$= \lim_{h\to 0}\frac{1}{\sqrt{3}}\tan^{-1}(\to-\infty) = -\frac{\pi}{2\sqrt{3}} \qquad \textbf{. . . (ii)}$$

From (i) and (ii), LHL $\neq$ RHL at $x = \pi/2$

$\Rightarrow$ Anti-derivative, $F(x)$ is discontinuous at $x = \pi/2$.

IN-CHAPTER EXERCISE-A

1. *Evaluate* $\int_{\log 2}^{\log 3} \frac{e^x}{1+e^x}\,dx.$

2. *Given the function :* $f(x) = \begin{cases} x^2 & ;0 \le x \le 1 \\ \sqrt{x} & ;1 \le x \le 2 \end{cases}$. *Evaluate* $\int_0^2 f(x)dx.$

3. *Evaluate the integral* $I = \int_0^2 |x-1|\,dx.$

4. *Evaluate :* $\int_0^{\pi/2} \frac{\sqrt{\cot x}}{\sqrt{\cot x} + \sqrt{\tan x}}\,dx$

5. *Show that* $\int_0^{\pi} x f(\sin x)\,dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,dx$

6. *Evaluate :* $\int_0^{\pi/2} \frac{x \sin x \cos x}{\cos^4 x + \sin^4 x}\,dx.$

7. *Show that* $\int_0^{\pi} \frac{x}{a^2\cos^2 x + b^2 \sin^2 x}\,dx = \frac{\pi^2}{2ab}$

8. *Prove that* $\int_a^b f(x)\,dx = (b-a)\int_0^1 f[(b-a)x+a]\,dx.$

9. *Compute the sum of the two integrals :* $\int_{-4}^{-5} e^{(x+5)^2}\,dx + 3\int_{1/3}^{2/3} e^{9(x-2/3)2}\,dx.$

10. *Evaluate :* $\int_{-\sqrt{2}}^{\sqrt{2}} \frac{2x^7+3x^6-10x^5-7x^3-12x^2+x+1}{x^2+2}\,dx.$

11. *Evaluate :* **(i)** $\int_0^{\pi/2} \sin^8 x\,dx$ **(ii)** $\int_0^{\pi/2} \sin^9 x \cos^7 x\,dx$ **(iii)** $\int_0^{\pi/2} \cos^9 x\,dx.$

12. *Show that :* $4 \le \int_1^3 \sqrt{3+x^3}\,dx \le 2\sqrt{30}.$

13. *Evaluate :* $\int_a^b e^x dx; \int_a^b \sin x\,dx$ using limit of a sum.

14. *Evaluate :* $\int_{-2}^{2} \frac{dx}{4+x^2}$ *directly as well as by the substitution* $x = 1/t$. *If answers do not tally, then explain why?*

PART-B : AREA Section - 4

4.1 Curve tracing

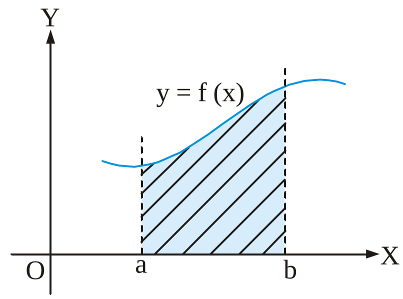
In order to find the area bounded by several curves, sometimes it is necessary to have an idea of the rough sketches of these curves. To find the approximate shape of a curve represented by the cartesian equation, the following steps are very useful.

1. Symmetry
   (a) If curve remains unaltered on replacing x by $-x$, then it is symmetrical about y-axis.
   (b) If curve remains unaltered on replacing y by $-y$, then it is symmetrical about x-axis.
2. Intersection with axes
   (a) To find points of intersection of the curve with x-axis, replace $y = 0$ in the equation of the curve and get corresponding values of x.
   (b) To find points of intersection of the curve with y-axis, replace $x = 0$ in the equation of the curve and get corresponding values of y.
3. The regions where curves does not exist
   (a) Find those values of x for which corresponding values of y do not exist.
   (b) Find intervals where $f(x)$ is positive.
4. Asymptotes
   (a) Observe where y approaches as x approaches $\pm\infty$.
   (b) If necessary, observe where x approaches as y approaches $\pm\infty$.
5. Find points of local maximum and local minimum
   Put $f'(x) = 0$ and find points of local maximum and minimum.

4.2 Important Results

1. If $f(x) \geq 0$ for all $x \in [a, b]$, then Area bounded by the curve $y = f(x)$, X-axis and the lines $x = a$ and $x = b$ is given by

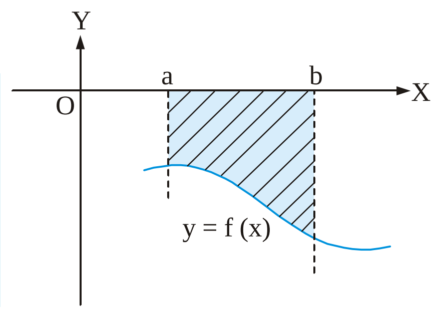
$$A = \int_a^b f(x)dx$$



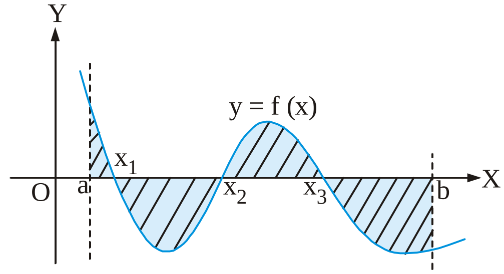
Note : The whole of the curve in the internal $[a, b]$ lies above X-axis.

2. If $f(x) \leq 0$ for all $x \in [a, b]$, then Area bounded by a curve $y = f(x)$. X-axis and the lines $x = a$ and $x = b$ is given by :

$$\text{Area} = \left| \int_a^b f(x)\,dx \right|$$



3. If the curve crosses X-axis one or more times in $[a, b]$, then the area bounded by the curve $y = f(x)$, X-axis and the lines $x = a$ and $x = b$ is calculated by considering the portions of the graph lying above X-axis and below X-axis separately. To calculate the area of the regions lying above X-axis, use result-1 and for the regions lying below X-axis, use result-2.

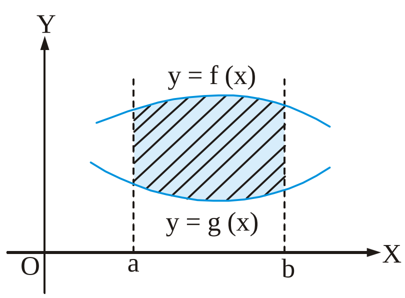


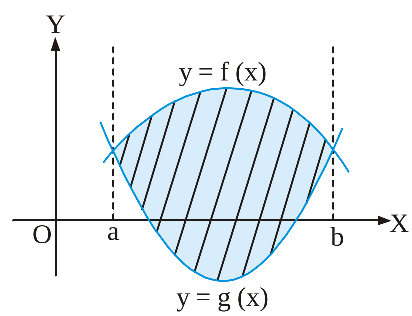
In the figure, the curve crosses X-axis at $x = x_1, x_2, x_3$.
Shaded area is given as follows :

$$A = \int_a^{x_1} f(x)\,dx + \left|\int_{x_1}^{x_2} f(x)\,dx\right| + \int_{x_2}^{x_3} f(x)\,dx + \left|\int_{x_3}^{b} f(x)\,dx\right|$$

4. Area bounded by two curves, $y = f(x)$ and $y = g(x)$, from above and below is given by :

$$\text{shaded area} = \int_a^b [f(x) - g(x)]\,dx$$





Note : The area is bounded from above by $y = f(x)$ and from below by $y = g(x)$.
The shaded area may be above or below X-axis.

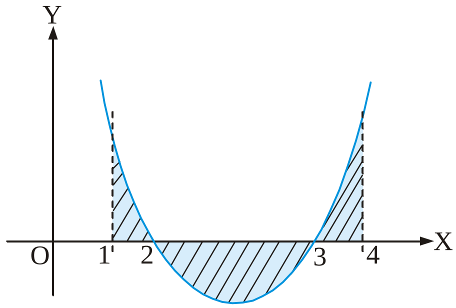
Illustration - 25 *The area bounded by the curve $y = x^2 - 5x + 6$, X-axis and the lines $x = 1$ and 4 is :*

(A) $\frac{9}{6}$ **(B)** $\frac{10}{6}$ **(C)** $\frac{11}{6}$ **(D)** *None of these*

SOLUTION : (C)

For $y = 0$, we get $x^2 + 5x + 6 = 0$
$\Rightarrow x = 2, 3$
Hence the curve crosses X-axis at $x = 2, 3$ in the interval $[1, 4]$.

$$\text{Bounded Area} = \int_1^2 y\,dx + \left|\int_2^3 y\,dx\right| + \int_3^4 y\,dx$$



$$\Rightarrow \quad A = \int_1^2 (x^2 - 5x + 6)\,dx + \left| \int_2^3 (x^2 - 5x + 6)\,dx \right| + \int_3^4 (x^2 - 5x + 6)\,dx$$

$$A_1 = \frac{2^3 - 1^3}{3} - 5\left(\frac{2^2 - 1^2}{2}\right) + 6\,(2-1) = \frac{5}{6}$$

$$A_2 = \frac{3^3 - 2^3}{3} - 5\left(\frac{3^2 - 2^2}{2}\right) + 6\,(3-2) = -\frac{1}{6}$$

$$A = \frac{4^3 - 3^3}{3} - 5\left(\frac{4^2 - 3^2}{2}\right) + 6\,(4-3) = \frac{5}{6}$$

$$\Rightarrow A = \frac{5}{6} + \left| -\frac{1}{6} \right| + \frac{5}{6} = \frac{11}{6} \text{ sq. units.}$$

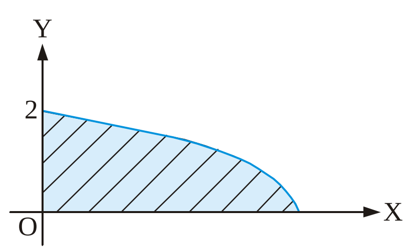
Illustration - 26 *The area bounded by the curve : $y = \sqrt{4-x}$, X-axis and Y-axis.*

(A) $\frac{8}{3}$ (B) $\frac{16}{3}$ (C) $\frac{32}{3}$ (D) *None of these*

SOLUTION : (B)

Trace the curve $y = \sqrt{4-x}$.

1. Put $y = 0$ in the given curve to get $x = 4$ as the point of intersecton with X-axis.
   Put $x = 0$ in the given curve to get $y = 2$ as the point of intersection with Y-axis.
2. For the curve, $y = \sqrt{4-x}$, $4 - x \geq 0$
   $\Rightarrow \quad x \leq 4$
   $\Rightarrow$ curve lies only to the left of $x = 4$ line.
3. As any y is positive, curve is above X-axis.



Using step 1 to 3, we can draw the rough sketch of

$y = \sqrt{4-x}$.

In figure,

Bounded area =

$$\int_0^4 \sqrt{4-x}\,dx = \left| \frac{-2}{3}(4-x)\sqrt{4-x} \right|_0^4 = \frac{16}{3} \text{ sq. units.}$$

Illustration - 27 *The area bounded by the curves $y = x^2$ and $x^2 + y^2 = 2$ above X-axis is :*

(A) $\frac{1}{6}+\frac{\pi}{4}$ (B) $\frac{2}{3}+\frac{\pi}{2}$ (C) $\frac{\pi}{4}+\frac{1}{6}$ (D) $\frac{1}{3}+\frac{\pi}{2}$

SOLUTION : (D)

Let us first find the points of intersection of curves.

Solving $y = x^2$ and $x^2 + y^2 = 2$ simultaneously,

we get :

$x^2 + x^4 = 2$

$\Rightarrow (x^2 - 1)(x^2 + 2) = 0$

$\Rightarrow x^2 = 1$ and $x^2 = -2$ [reject]

$\Rightarrow x = \pm 1$

$\Rightarrow A = (-1, 0)$ and $B = (1, 1)$

Shaded Area = $\int_{-1}^{+1}\left(\sqrt{2-x^2} - x^2\right) dx$

$= \int_{-1}^{+1}\sqrt{2^x - x^2}\, dx - \int_{-1}^{+1} x^2\, dx$

$= 2\int_0^1 \sqrt{2-x^2}\, dx - 2\int_0^1 x^2\, dx$

$= 2\left|\frac{x}{2}\sqrt{2-x^2} + \frac{2}{2}\sin^{-1}\frac{x}{\sqrt{2}}\right|_0^1 - 2\left(\frac{1}{3}\right)$

$= 2\left(\frac{1}{2}+\frac{\pi}{4}\right) - \frac{2}{3} = \frac{1}{3}+\frac{\pi}{2}$ sq. units.

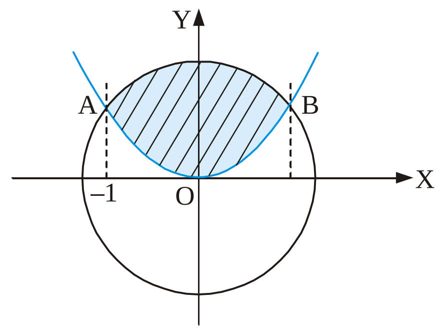


Illustration - 28 *The area bounded by $y = x^2 - 4$ and $x + y = 2$ is :*

(A) $\frac{75}{6}$ (B) $\frac{100}{6}$ (C) $\frac{125}{6}$ (D) $\frac{150}{6}$

SOLUTION : (C)

After drawing the figure, let us find the points of intersection of

$y = x^2 - 4$ and $x + y = 2$.

$\Rightarrow x + x^2 - 4 = 2 \Rightarrow x^2 + x - 6 = 0$

$\Rightarrow (x + 3)(x - 2) = 0$

$\Rightarrow x = -3, 2$

$\Rightarrow A \equiv (-3, 5)$ and $B \equiv (2, 0)$

Shaded area, $= \int_{-3}^{2}\left[(2-x)-(x^2-4)\right] dx$

$= \int_{-3}^{2}(2-x)\, dx - \int_{-3}^{2}(x^2-4)\, dx$

$= \left|2x - \frac{x^2}{2}\right|_{-3}^{2} - \left|\frac{x^3}{3} - 4x\right|_{-3}^{2}$

$= 2\times 5 - \frac{1}{2}(4-9) - \frac{1}{3}(8+27) + 4(5) = \frac{125}{6}$

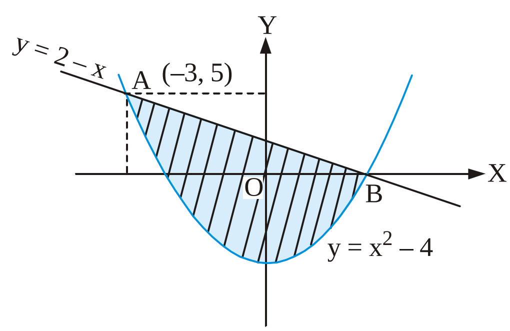


Illustration - 29 *The area bounded by the circle $x^2 + y^2 = a^2$ is :*

(A) $\frac{\pi a^2}{4}$ (B) $\frac{\pi a^2}{2}$ (C) πa^2 (D) $2\pi a^2$

SOLUTION : (C)

$x^2 + y^2 = a^2 \quad \Rightarrow \quad y = \pm\sqrt{a^2 - x^2}$

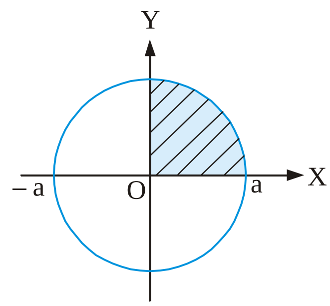
Equation of semicircle above X-axis is $y = +\sqrt{a^2 - x^2}$

Area of circle = 4 (shaded area)

$$= 4\int_0^a \sqrt{a^2 - x^2}\, dx$$

$$= 4\left| \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} \right|_0^a$$

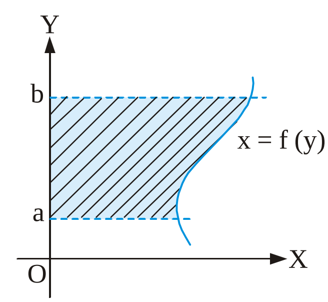
$$= 4\frac{a^2}{2}\left(\frac{\pi}{2}\right) = \pi a^2$$



4.3 Important Results (Contd.....)

5. If $f(y) \geq 0$ for all $y \in [a, b]$, then the Area bounded by a curve $x = f(y)$, Y-axis and the lines $y = a$ and $y = b$ is given by

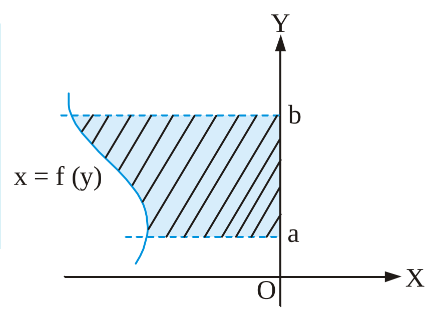
$$\text{Area} = \int_a^b f(y)\, dy$$



Note : The whole of the curve in the interval $[a, b]$ lies on right of Y-axis.

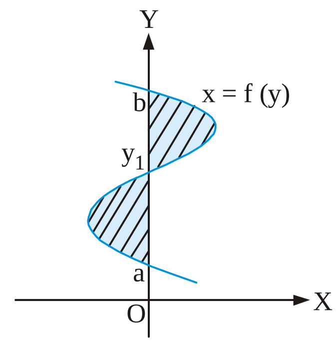
6. If $f(y) \leq 0$ for all $y \in [a, b]$, then the Area bounded by a curve $x = f(y)$, Y-axis and the lines $y = a$ and $y = b$ is given by

$$\text{Area} = \left| \int_a^b f(y)\, dy \right|$$



Note : The whole of the curve in the interval $[a, b]$ lies on left of Y-axis.

7. If the curve crosses Y-axis one or more times in $[a, b]$, then the area bounded by the curve $x = f(y)$, Y-axis and the lines $y = a$ and $y = b$ is calculated by considering the portions of the graph lying on the right side and the left side of the Y-axis separately. To calculate the area of the regions lying on right-hand side of the Y-axis, use result-5 and for the regions lying on left-hand side, use result - 6.



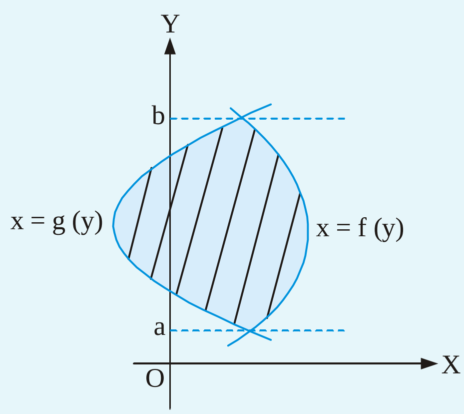
In the figure, the curve crosses Y-axis at $y = y_1$.

Shaded area is given as follows : $A = \left| \int_a^{y_1} f(x)\, dy \right| + \int_{y_1}^{b} f(y)\, dy$

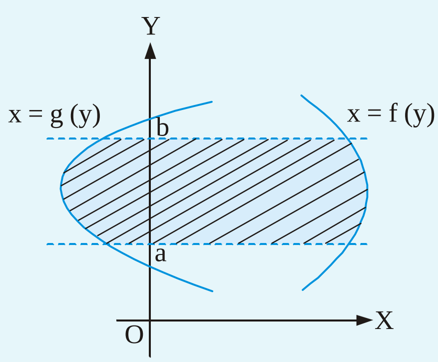
8. Area bounded by two curves, $x = f(y)$ and $x = g(y)$, from right and left respectively, is given by :

Shaded area = $\int_a^b [f(y) - g(y)]\, dy$

Note : The area is bounded from right by $x = f(y)$ and from left by $x = g(y)$.



The shaded area may be on right or left side of the Y-axis.



9. If the equations of the curves are expressed in parametric form, then the area bounded can not be found by direct application of the result 1 to 8.
Let the two curves in parametric form are

$x = f(t)$ . . . (i) and $y = g(t)$ . . .(ii)

To find the bounded area by curves, try to eliminate parameter t in equations (i) and (ii) to express y in terms of x (or x in terms of y). If it is possible to eliminate t, then the required area can be obtained by using the results 1 to 8.
If it is not possible to eliminate t, then the required area can be obtained by using the following formula :

$$\text{Area} = \int_a^b y\,dx = \int_a^b y\frac{dx}{dt}\,dt = \int_{t_1}^{t_2} g(t)\,f'(t)\,dt$$ where t_1 and t_2 are given by $f(t_1) = a$ and $f(t_2) = b$.

Illustration - 30 *The area bounded by the curves $x^2 + y^2 = 4a^2$ and $y^2 = 3ax$ is :*

(A) $\left(\frac{1}{\sqrt{3}}+\frac{4\pi}{3}\right)a^2$ (B) $\left(\frac{1}{2\sqrt{3}}+\frac{2\pi}{3}\right)a^2$ (C) $\left(\frac{1}{\sqrt{3}}+\frac{2\pi}{3}\right)a^2$ (D) $\left(\frac{1}{2\sqrt{3}}+\frac{4\pi}{3}\right)a^2$

SOLUTION : (A)

The points of intersection A & B can be calculated.
by solving $x^2 + y^2 = 4a^2$ and $y^2 = 3ax$.

$$\Rightarrow \left(\frac{y^2}{3a}\right)^2 + y^2 = 4a^2$$

$$\Rightarrow y^4 + 9a^2y^2 - 36a^4 = 0$$

$$\Rightarrow (y^2 - 3a^2)(y^2 + 12a^2) = 0$$

$\Rightarrow y^2 = 3a^2$ or $y^2 = -12a^2$ (reject)

$\Rightarrow y^2 = 3a^2$ $\Rightarrow$ $y = \pm\sqrt{3}a$

The equation of right half of

$x^2 + y^2 = 4a^2$ is $x = \sqrt{4a^2 - y^2}$

$$\text{Shaded area} = \int_{-\sqrt{3}a}^{\sqrt{3}a}\left(\sqrt{4a^2-y^2} - \frac{y^2}{3a}\right)dy$$

$$= 2\int_0^{\sqrt{3}a}\left(\sqrt{4a^2-y^2} - \frac{y^2}{3a}\right)dy$$

[using property - 8]

$$= 2\left|\frac{y}{2}\sqrt{4a^2-y^2} + \frac{4a^2}{2}\sin^{-1}\frac{y}{2a}\right|_0^{\sqrt{3}a} - \frac{2}{3a}\left|\frac{y^3}{3}\right|_0^{\sqrt{3}a}$$

$$= \sqrt{3a^2} + 4a^2\frac{\pi}{3} - \frac{2}{9a}3\sqrt{3}a^3$$

$$= \left(\frac{1}{\sqrt{3}} + \frac{4\pi}{3}\right)a^2$$

Alternative Method :

shaded area = 2 × (area above X-axis)

x-coordinate of $A = \frac{y^2}{3a} = \frac{3a^2}{3a} = a$

The given curves area

$y = \pm\sqrt{3ax}$ and $y = \pm\sqrt{4a^2 - x^2}$

The above the X-axis, the equations of the parabola and the circle are $\sqrt{3ax}$ and $y = \sqrt{4a^2 - x^2}$ respectively.

$\Rightarrow$ Shaded area

$$= 2\left[\int_0^a \sqrt{3ax}\,dx + \int_a^{2a} \sqrt{4a^2 - x^2}\,dx\right]$$

Solve it yourself to get the answer.

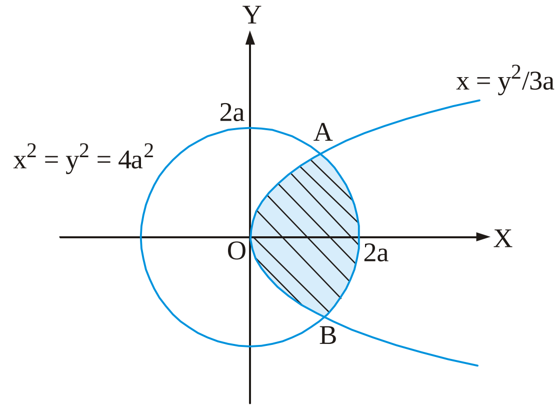


Illustration - 31 *The area bounded by the curves : $y^2 = 4a(x + a)$ and $y^2 = 4b(b - x)$ is :*

(A) $\frac{2}{3}(a+b)\sqrt{4ab}$ **(B)** $\frac{4}{3}(a+b)\sqrt{4ab}$ **(C)** $\frac{8}{3}(a+b)\sqrt{4ab}$ **(D)** *None of these*

SOLUTION : (B)

The two curves are :

$y^2 = 4a(x + a)$ ... (i)

and $y^2 = 4b(b - x)$ ... (ii)

Solving $y^2 = 4a(x + a)$ and $y^2 = 4b(b - x)$ simultaneously,

we get the coordinates of A and B.

Replacing values of x from (ii) and (i), we get :

$$y^2 = 4a\left(b - \frac{y^2}{4b} + a\right)$$

$\Rightarrow$ $y = \pm\sqrt{4ab}$ and $x = b - a$.

$\Rightarrow A \equiv (b - a, \sqrt{4ab})$ and $B \equiv (b - a, -\sqrt{4ab})$

$$\text{shaded area} = \int_{-\sqrt{4ab}}^{\sqrt{4ab}} \left[\left(b - \frac{y^2}{4b}\right) - \left(\frac{y^2}{4a} - a\right)\right] dy$$

$$\Rightarrow A = 2(a+b)\sqrt{4ab} - \int_0^{\sqrt{4ab}} \left(\frac{y^2}{2b} + \frac{y^2}{2a}\right) dy$$

[using property - 8]

$$\Rightarrow A = 2(a+b)\sqrt{4ab} - \frac{1}{2}\left[\frac{4ab\sqrt{4ab}}{3b} + \frac{4ab\sqrt{4ab}}{3a}\right]$$

$$\Rightarrow A = \frac{4}{3}(a+b)\sqrt{4ab}$$

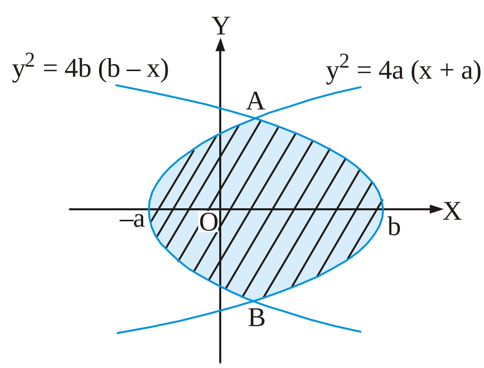


Illustration - 32 *The area bounded by the hyperbola : $x^2 - y^2 = a^2$ and the line $x = 2a$ is :*

(A) $\sqrt{3}a^2 - a^2\log\left(2+\sqrt{3}\right)$ **(B)** $2\sqrt{3}a^2 - a^2\log\left(2+\sqrt{3}\right)$

(C) $\sqrt{3}a^2 - a^2\log\left(2-\sqrt{3}\right)$ **(D)** $2\sqrt{3}a^2 - a^2\log\left(2-\sqrt{3}\right)$

SOLUTION : (B)

Shaded area $= 2 \times$ (Area of the portion above X- axis)

The equation of the curve above x-axis is :

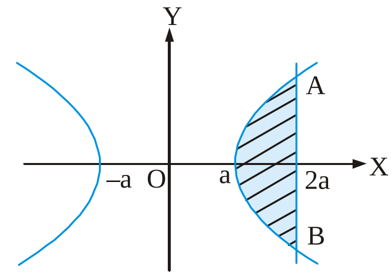
$$y = \sqrt{x^2 - a^2}$$

$$\Rightarrow \text{ required area } (A) = 2\int_a^{2a}\sqrt{x^2 - a^2}\,dx$$

$$\Rightarrow A = 2\left|\frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right|\right|_a^{2a}$$

$$\Rightarrow A = 2\sqrt{3}\,a^2 - a^2\log\left|(2a + \sqrt{3}a)\right| + a^2\log a$$

$$\Rightarrow A = 2\sqrt{3}\,a^2 - a^2\log(2 + \sqrt{3}).$$



Alternative Method :

$$\text{Area } (A) = \int_{y_B}^{y_A}\left(2a - \sqrt{a^2 + y^2}\right)dy$$

$$\Rightarrow A = \int_{-\sqrt{3}a}^{\sqrt{3}a}\left(2a - \sqrt{a^2 + y^2}\right)dy$$

Solve it yourself to confirm.

Illustration - 33 *The area bounded by the curves : $x^2 + y^2 = 25$, $4y = |4 - x^2|$ and $x = 0$ in the first quadrant is :*

(A) $2 + \frac{25}{2}\sin^{-1}\left(\frac{4}{5}\right)$ **(B)** $2 + \frac{25}{4}\sin^{-1}\left(\frac{4}{5}\right)$

(C) $1 + \frac{25}{2}\sin^{-1}\left(\frac{4}{5}\right)$ **(D)** $1 + \frac{25}{4}\sin^{-1}\left(\frac{4}{5}\right)$

SOLUTION : (A)

First of all find the coordinates of points of intersec -tion A by solving the equations of two given curves :

$\Rightarrow x^2 + y^2 = 25$ and $4y = |4 - x^2|$

$$\Rightarrow x^2 + \frac{(4 - x^2)^2}{16} = 25$$

$\Rightarrow (x^2 - 4)^2 + 16x^2 = 400.$

$\Rightarrow (x^2 + 4)^2 = 400 \quad \Rightarrow \quad x^2 = 16$

$\Rightarrow x = \pm 4 \quad \Rightarrow \quad y = \frac{|4 - x^2|}{4} = 3$

$\Rightarrow$ Coordinates of point are $A \equiv (4, 3)$

$$\text{Shaded area} = \int_0^4\left[\sqrt{25 - x^2} - \frac{|4 - x^2|}{4}\right]dx$$

$$\Rightarrow A = \int_0^4\sqrt{25 - x^2}\,dx - \frac{1}{4}\int_0^4|4 - x^2|\,dx \quad \ldots\text{(i)}$$

Let $I = \frac{1}{4}\left|4 - x^2\right| dx$

$$\Rightarrow \quad I = \frac{1}{4}\int_0^2 (4 - x^2)\,dx + \frac{1}{4}\int_2^4 (x^2 - 4)\,dx$$

$$\Rightarrow \quad I = \frac{1}{4}\left(8 - \frac{8}{3}\right) - \frac{1}{4}\left(\frac{56}{3} - 8\right)$$

$\Rightarrow \quad I = 4$

On substituting the value of I in (i), we get :

$$A = \int_0^4 \sqrt{25 - x^2}\,dx - 4$$

$$\Rightarrow A = \left|\frac{x}{2}\sqrt{25 - x^2} + \frac{25}{2}\sin^{-1}\frac{x}{5}\right|_0^4 - 4$$

$$\Rightarrow A = 6 + \frac{25}{2}\sin^{-1}\frac{4}{5} - 4 = 2 + \frac{25}{2}\sin^{-1}\frac{4}{5}$$

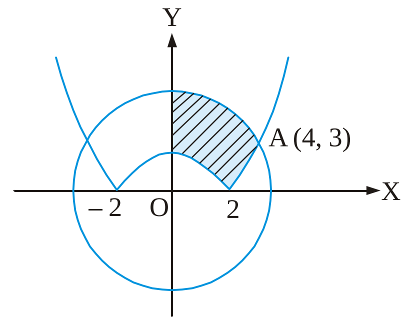


Illustration - 34 *The area enclosed by the loop in the curve : $4y^2 = 4x^2 - x^3$ is :*

(A) $\frac{32}{15}$ (B) $\frac{64}{15}$ (C) $\frac{128}{15}$ (D) $\frac{256}{15}$

SOLUTION : (C)

The given curve is : $4y^2 = 4x^2 - x^3$

To draw the rough sketch of the given curve, consider the following steps :

(i) On replacing y by $-y$, there is no change in function. It means the graph is symmetric about Y-axis.

(ii) For $x = 4$, $y = 0$ and for $x = 0$, $y = 0$.

(iii) In the given curve, LHS is positive for all values of y.

$\Rightarrow$ RHS ≥ 0 $\Rightarrow$ $x^2(1 - x/4) \geq 0$

$\Rightarrow$ $x \leq 4$.

Hence the curve lies to the left of $x = 4$.

(iv) As $x \to -\infty$, $y \to \pm\infty$

(v) Points of maximum/minimum :

$$8y\frac{dy}{dx} = 8x - 3x^2$$

$$\frac{dy}{dx} = 0 \quad \Rightarrow \quad x = 0, \frac{8}{3}$$

At $x = 0$, derivative is not defined.

By checking for $\frac{d^2y}{dx^2}$, $x = \frac{8}{3}$ is a point of local maximum (above X-axis).

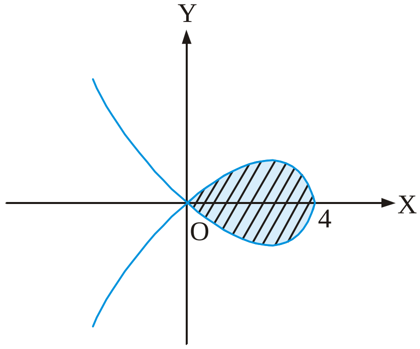
From graph,

Shaded area (A) = 2 × (area of portion above X-axis)

$$\Rightarrow \quad A = 2\int_0^4 \frac{x}{2}\sqrt{4 - x}\,dx = \int_0^4 x\sqrt{4 - x}\,dx$$

$$\Rightarrow \quad A = \int_0^4 (4 - x)\sqrt{4 - (4 - x)}\,dx$$

[using property - 4]



$$\Rightarrow A = \int_0^4 (4-x)\sqrt{x}\, dx$$

$$\Rightarrow A = 4\left|\frac{2}{3}x\sqrt{x}\right|_0^4 - \left|\frac{2}{5}x^2\sqrt{x}\right|_0^4$$

$$\Rightarrow A = \frac{128}{15} \text{ sq. units.}$$

Illustration - 35 *The area bounded by the parabola y = x^2, X-axis and the tangent to the parabola at* (1, 1) *is* :

(A) $\frac{1}{4}$ **(B)** $\frac{1}{6}$ **(C)** $\frac{1}{9}$ **(D)** $\frac{1}{12}$

SOLUTION : (D)

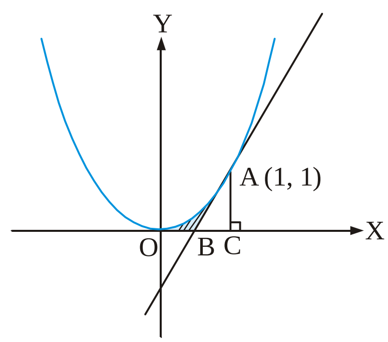
The given curve is $y = x^2$. Equation of tangent at $A = (1, 1)$ is :

$$y - 1 = \left.\frac{dy}{dx}\right]_{x=1} \cdot (x-1) \qquad \text{[using : } y - y_1 = m\,(x - x_1)\text{]}$$

$\Rightarrow \quad y - 1 = 2\,(x - 1) \qquad \Rightarrow \qquad y = 2x - 1 \qquad$ . . . (i)

The point of intersection of (i) with X-axis is $B \equiv (1/2, 0)$.

Shaded area = area ($OACO$) – area (ABC)



$$\Rightarrow \quad \text{area} = \int_0^1 x^2 dx - \int_{1/2}^1 (2x-1)\, dx$$

$$\Rightarrow \quad \text{area} = \frac{1}{3}\left[1 - \frac{1}{4} - (1 - 1/2)\right]$$

$$\Rightarrow \quad \text{area} = \frac{1}{12}$$

Illustration - 36 *The area between the curves $y = 2x^4 - x^2$, the x-axis and the ordinates of two minima of the curve is :*

(A) $\frac{7}{240}$ **(B)** $\frac{7}{120}$ **(C)** $\frac{7}{60}$ **(D)** *None of these*

SOLUTION : (B)

Using the curve tracing steps , draw the rough sketch of the functions $y = 2x^4 - x^2$.
Following are the properties of the curve which can be used to draw its rough sketch.

(i) The curve is symmetrical about y-axis.

(ii) Point of intersection with x-axis are $x = 0, x = \pm\frac{1}{\sqrt{2}}$. Only point of intersection with y-axis is $y = 0$.

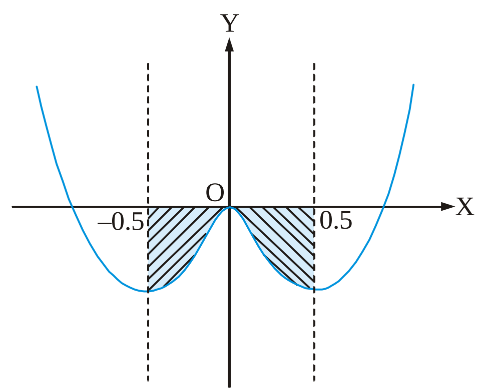
(iii) For $x \in \left(-\infty, -\frac{1}{\sqrt{2}}\right) \cup \left(\frac{1}{\sqrt{2}}, \infty\right)$, $y > 0$ i.e. curve lies above x axis and in the other intervals it lies below x-axis.

(iv) Put $\frac{dy}{dx} = 0$ to get $x = \pm 1/2$ as the points of local minimum.

On plotting the above information on graph, we get the rough sketch of the graph. The shaded area in the graph is the required area

$$\text{Required Area} = 2\left|\int_0^{1/2}\left(2x^4 - x^2\right)dx\right|$$

$$= 2\left|\left[\frac{2x^5}{5} - \frac{x^3}{3}\right]_0^{1/2}\right| = \frac{7}{120}$$



FOR REMAINING QUESTIONS ATTEMPT IN-CHAPTER EXERCISE-B

NOW ATTEMPT OBJECTIVE WORKSHEET BEFORE PROCEEDING AHEAD IN THIS EBOOK

THINGS TO REMEMBER

1. The relationship between the definite integral $\int_a^b f(x)\,dx$ and the indefinite integral $F(x)$ is :

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

2. Properties in Definite Integral

PROPERTY - 1 :

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

PROPERTY - 2 :

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

PROPERTY - 3 :

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$$

PROPERTY - 4 :

$$\int_0^a f(x)dx = \int_0^a f(a-x)\,dx$$

PROPERTY - 5 :

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$$

PROPERTY - 6 :

$$\int_0^{2a} f(x)\,dx = 0 \quad \text{if} \quad (2a-x) = -f(x) \quad ; \quad \int_0^{2a} f(x)dx = 2\int_0^a f(x)dx \text{ if } f(2a-x) = f(x)$$

PROPERTY - 7 :

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

PROPERTY - 8 :

$$\int_{-a}^{a} f(x)\,dx = \int_0^a [f(x) + f(-x)]\,dx$$

$= 2\int_0^a f(x)dx$ if $f(x)$ is even i.e. $f(-x) = f(x)$

$= 0$ if $f(x)$ is odd i.e $f(-x) = -f(x)$

PROPERTY - 9 :

$$\int_0^{nT} f(x)\,dx = n\int_0^{T} f(x)\,dx$$ where $f(x)$ is a periodic function with period T and n is an integer.

PROPERTY - 10 :

$$\int_0^{\infty} f(x)\,dx = \lim_{n\to\infty}\int_0^{n} f(x)\,dx \quad ; \quad \int_{-\infty}^{a} f(x)\,dx = \lim_{n\to-\infty}\int_n^{a} f(x)\,dx$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_0^{\infty} f(x)\,dx = \lim_{a\to-\infty}\int_a^{0} f(x)\,dx + \lim_{b\to\infty}\int_0^{b} f(x)\,dx$$

PROPERTY - 11 : $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$.

PROPERTY - 12 :

If the function $f(x)$ and $g(x)$ are defined on $[a, b]$ and differentiable at all points x.

$\in [f(a), g(b)]$, then $$\frac{d}{dx}\left[\int_{f(x)}^{g(x)} h(t)\,dt\right] = h[g(x)]\,g'(x) - h[f(x)]\,f'(x)$$

PROPERTY - 13 :

If $f(x) \ge g(x)$ for all $x \in [a, b]$, then $\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$.

Put $g(x) = 0$ for all $x \in [a, b]$ in above property to get another useful property, i.e.

If $f(x) \ge 0$ for all $x\ \varepsilon\ [a, b]$, then $\int_a^b f(x)dx \ge 0$.

3. The definite integral of a function *f* (*x*) in the interval [*a*, *b*] can be defined as :

$$\int_a^b f(x)\,dx = \lim_{\substack{n\to\infty \\ h\to 0}} h[f(a+h) + f(a+2h) + \ldots.. + f(a+nh)]$$ where $nh = b - a$.

$$= \lim_{\substack{n\to\infty \\ h\to 0}} h\sum_{r=1}^{n} f(a+rh)$$ where $nh = b - a$.

4. Estimation of a Definite Integral

If $f(x)$ is a function defined in the interval $[a, b]$ then :

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$

where m is the least and M is the greatest value of the function $f(x)$ in the interval $[a, b]$.

5. Mean value theorem of definite Integrals

If the function $f(x)$ is continuous in the interval $[a, b]$, then :

$$\int_a^b f(x)\,dx = f(c)(b-a), \qquad \text{where } a < c < b.$$

6. Two useful Formulae

(i) If n be a positive integer, then :

$$\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx$$

$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\cdots\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}, \qquad \text{when } n \text{ is even}$$

$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\cdots\frac{4}{5}\cdot\frac{2}{3}\cdot 1, \qquad \text{when } n \text{ is odd}$$

(ii)

$$\int_0^{\pi/2} \sin^m x\cos^n x\,dx = \int_0^{\pi/2} \sin^n x\cos^m dx$$

$$= \frac{(m-1)\,.\,(m-3)\,\ldots\ldots\,(1 \text{ or } 2)\,(n-1)\,.\,(n-3)\,\ldots\ldots\,(1 \text{ or } 2)}{(m+n)\,.\,(m+n-2)\,\ldots\ldots\,(1 \text{ or } 2)}\,\frac{\pi}{2},$$

when both m and n $\in$ even integer

$$= \frac{(m-1)\,.\,(m-3)\,\ldots\ldots\,(1 \text{ or } 2)\,(n-1)\,.\,(n-3)\,\ldots\ldots\,(1 \text{ or } 2)}{(m+n)\,.\,(m+n-2)\,\ldots\ldots\,(1 \text{ or } 2)},$$

when both m and n $\in$ even integer.

7. Curve tracing

In order to find the area bounded by several curves, sometimes it is necessary to have an idea of the rough sketches of these curves. To find the approximate shape of a curve represented by the cartesian equation, the following steps are very useful.

1. Symmetry
   - (a) If curve remains unaltered on replacing x by $-x$, then it is symmetrical about y-axis.
   - (b) If curve remains unaltered on replacing y by $-y$, then it is symmetrical about x-axis.
2. Intersection with axes
   - (a) To find points of intersection of the curve with x-axis, replace $y = 0$ in the equation of the curve and get corresponding values of x.
   - (b) To find points of intersection of the curve with y-axis, replace $x = 0$ in the equation of the curve and get corresponding values of y.

3. The regions where curves does not exist
   (a) Find those values of x for which corresponding values of y do not exist.
   (b) Find intervals where $f(x)$ is positive.

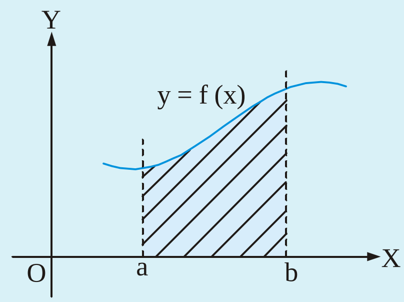
4. Asymptotes
   (a) Observe where y approaches as x approaches $\pm\infty$.
   (b) If necessary, observe where x approaches as y approaches $\pm\infty$.

5. Find points of local maximum and local minimum
   Put $f'(x) = 0$ and find points of local maximum and minimum.

8. Important Results

I. If $f(x) \geq 0$ for all $x \in [a, b]$, then Area bounded by the curve $y = f(x)$, X-axis and the lines x = a and $x = b$ is given by

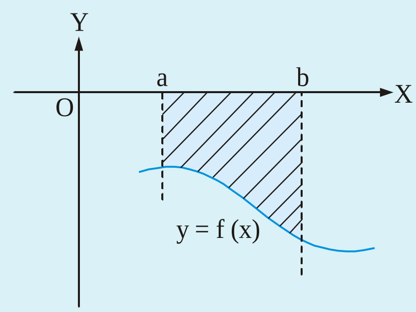
$$A = \int_a^b f(x)dx$$



Note : The whole of the curve in the internal $[a, b]$ lies above X-axis.

II. If $f(x) \leq 0$ for all $x \in [a, b]$, then Area bounded by a curve $y = f(x)$. X-axis and the lines $x = a$ and $x = b$ is given by :

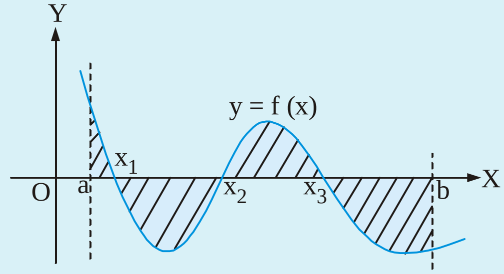
$$\text{Area} = \left| \int_a^b f(x)\, dx \right|$$



Note : The whole of the curve in the interval $[a, b]$ lies below X-axis.

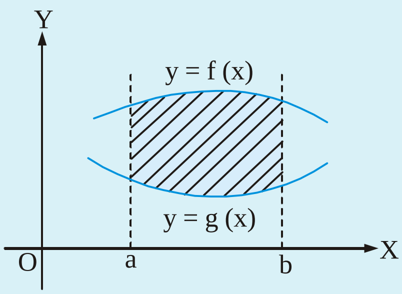
III. If the curve crosses X-axis one or more times in $[a, b]$, then the area bounded by the curve $y = f(x)$, X-axis and the lines $x = a$ and $x = b$ is calculated by considering the portions of the graph lying above X-axis and below X-axis separately. To calculate the area of the regions lying above X-axis, use result-1 and for the regions lying below X-axis, use result-2.
In the figure, the curve crosses X-axis at $x = x_1, x_2, x_3$.
Shaded area is given as follows :

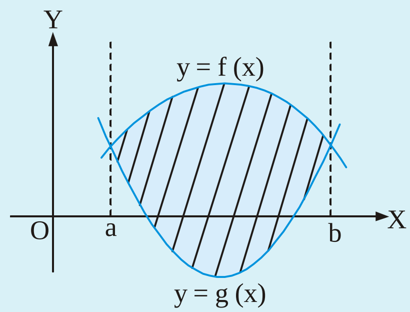
$$A = \int_{a}^{x_1} f(x)\,dx + \left| \int_{x_1}^{x_2} f(x)\,dx \right| + \int_{x_2}^{x_3} f(x)\,dx + \left| \int_{x_3}^{b} f(x)\,dx \right|$$



4. Area bounded by two curves, $y = f(x)$ and $y = g(x)$, from above and below is given by :

shaded area = $\int_{a}^{b} [f(x) - g(x)]\,dx$

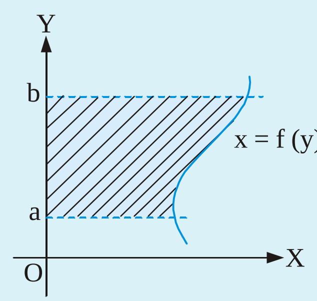




5. If $f(y) \geq 0$ for all $y \in [a, b]$, then the Area bounded by a curve $x = f(y)$, Y-axis and the lines $y = a$ and $y = b$ is given by

Area = $\int_{a}^{b} f(y)\,dy$

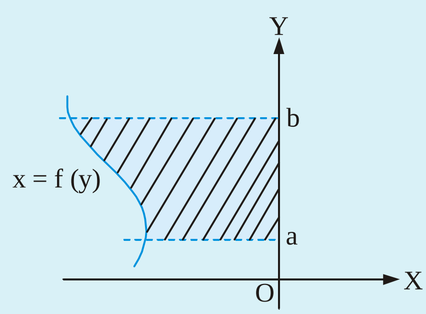
Note : The whole of the curve in the interval $[a, b]$ lies on right of Y-axis.



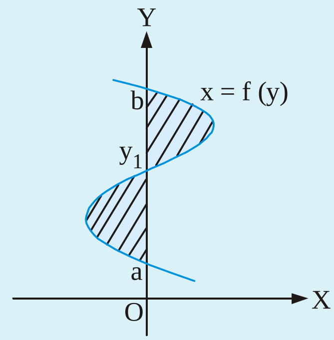
6. If $f(y) \leq 0$ for all $y \in [a, b]$, then the Area bounded by a curve $x = f(y)$, Y-axis and the lines $y = a$ and $y = b$ is given by

Area = $\left| \int_{a}^{b} f(y)\,dy \right|$

Note : The whole of the curve in the interval $[a, b]$ lies on left of Y-axis.



7. If the curve crosses Y-axis one or more times in $[a, b]$, then the area bounded by the curve $x = f(y)$, Y-axis and the lines $y = a$ and $y = b$ is calculated by considering the portions of the graph lying on the right side and the left side of the Y-axis separately. To calculate the area of the regions lying on right-hand side of the Y-axis, use result-5 and for the regions lying on left-hand side, use result - 6.
In the figure, the curve crosses Y-axis at $y = y_1$.
Shaded area is given as follows :

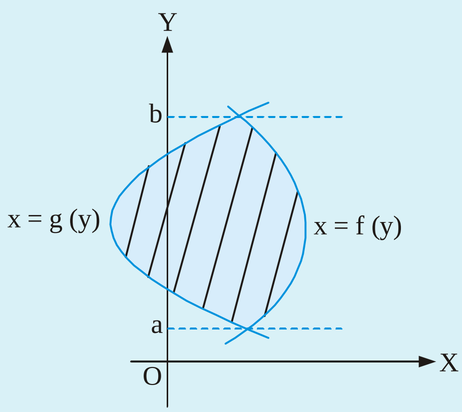


$$A = \left| \int_a^{y_1} f(x)\, dy \right| + \int_{y_1}^{b} f(y)\, dy$$

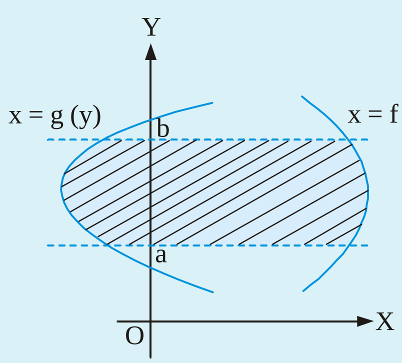
8. Area bounded by two curves, $x = f(y)$ and $x = g(y)$, from right and left respectively, is given by :

$$\text{Shaded area} = \int_a^b [f(y) - g(y)]\, dy$$

Note : The area is bounded from right by $x = f(y)$ and from left by $x = g(y)$.



The shaded area may be on right or left side of the Y-axis.



ANSWERS TO IN-CHAPTER EXERCISES - A

A

1. $ln\,\frac{4}{3}$ **2.** $\frac{1}{3}(4\sqrt{2}-1)$ **3.** 1 **4.** $\frac{\pi}{4}$ **6.** $\frac{\pi^2}{16}$ **9.** 0

10. $\frac{\pi}{2\sqrt{2}}-\frac{16\sqrt{2}}{5}$ **11.** (i) $\frac{35\pi}{256}$ (ii) $\frac{1}{560}$ (iii) $\frac{128}{315}$. **14.** Discontinuity at x = 0.

SOLUTIONS - IN-CHAPTER EXERCISE-A

1. $$\int_{\ln 2}^{\ln 3}\frac{e^x\,dx}{1+e^x}=\int_{\ln 2}^{\ln 3}\frac{d(1+e^x)}{1+e^x}\,dx=\ln\left|1+e^x\right|_{\ln 2}^{\ln 3}=\ln\frac{4}{3}$$

2. $$\int_0^2 f(x)\,dx=\int_0^1 x^2dx+\int_1^2\sqrt{x}dx=\left.\frac{x^3}{3}\right|_0^1+\left.\frac{2}{3}x\sqrt{x}\right|_1^2=\frac{1}{3}\left(4\sqrt{2}-1\right).$$

3. $$I=\int_0^2|x-1|\,dx=\int_0^1(1-x)\,dx+\int_1^2(x-1)\,dx=\left.\left(x-\frac{x^2}{2}\right)\right|_0^1+\left.\left(\frac{x^2}{2}-x\right)\right|_1^2=1.$$

4. $$I=\int_0^{\pi/2}\frac{\sqrt{\cot x}}{\sqrt{\cot x}+\sqrt{\tan x}}dx=\int_0^{\pi/2}\frac{\sqrt{\cot\left(\frac{\pi}{2}-x\right)}}{\sqrt{\cot\left(\frac{\pi}{2}-x\right)}+\sqrt{\tan\left(\frac{\pi}{2}-x\right)}}dx=\int_0^{\pi/2}\frac{\sqrt{\tan x}}{\sqrt{\tan x}+\sqrt{\cot x}}$$

On adding, $2I=\int_0^{\pi/2}dx\ \Rightarrow\ I=\frac{\pi}{4}$

5. $$I=\int_0^{\pi}x\,f(\sin x)\,dx=\int_0^{\pi}(\pi-x)\,f(\sin(\pi-x)\,dx=\pi\int_0^{\pi}f(\sin x)\,dx-I$$

$$\Rightarrow 2I=\pi\int_0^{\pi}f(\sin x)\,dx\Rightarrow I=\frac{\pi}{2}\int_0^{\pi}f(\sin x)\,dx.$$

6. $$I=\int_0^{\pi/2}\frac{\left(\frac{\pi}{2}-x\right)\sin\left(\frac{\pi}{2}-x\right)\cos\left(\frac{\pi}{2}-x\right)}{\cos^4\left(\frac{\pi}{2}-x\right)+\sin^4\left(\frac{\pi}{2}-x\right)}dx=\int_0^{\pi/2}\frac{\left(\frac{\pi}{2}-x\right)\cos x\sin x}{\sin^4 x+\cos^4 x}\cdot dx$$

$$\Rightarrow\quad I=\int_0^{\pi/2}\frac{\cos x\sin x\,dx}{\sin^4 x+\cos^4 x}-I\quad\Rightarrow 2I=\frac{\pi}{2}\int_0^{\pi/2}\frac{\cos x\sin x\,dx}{\sin^4 x+\cos^4 x}$$

$\Rightarrow \quad I = \dfrac{\pi}{4}\int_0^{\pi/2}\dfrac{\tan x \sec^2 x\, dx}{1+\tan^4 x}$ · let $\tan^2 x = t \Rightarrow 2\tan x \sec^2 x\, dx = dt$

$\Rightarrow \quad I = \dfrac{\pi}{8}\int_0^{\infty}\dfrac{dt}{1+t^2} = \dfrac{\pi}{8}\left.\tan^{-1} t\right|_0^{\infty} = \dfrac{\pi}{8}\left[\dfrac{\pi}{2}\right] = \dfrac{\pi^2}{16}$

7. $I = \int_0^{\pi}\dfrac{(\pi - x)\, dx}{a^2\cos^2 x + b^2 \sin^2 x} = \pi\int_0^{\pi}\dfrac{dx}{a^2\cos^2 x + b^2\sin^2 x} - I$

$\Rightarrow \quad I = \dfrac{\pi}{2}\int_0^{\pi}\dfrac{dx}{a^2\cos^2 x + b^2\sin^2 x} = \pi\int_0^{\pi/2}\dfrac{dx}{a^2\cos^2 x + b^2\sin^2 x} = \pi\int_0^{\infty}\dfrac{dt}{a^2 + b^2 t^2}$ $(\tan x = t)$

$= \dfrac{\pi}{b^2}\int_0^{\infty}\dfrac{dt}{\left(\dfrac{a}{b}\right)^2 + t^2} = \dfrac{\pi}{ab}\left.\tan^{-1}\dfrac{bt}{a}\right|_0^{\infty} = \dfrac{\pi}{ab}\left[\dfrac{\pi}{2}\right] = \dfrac{\pi^2}{2ab}$

8. LHS. $= \int_a^b f(x)\, dx$ Put $x = (b-a)t + a \Rightarrow dx = (b-a)\, dt$

$\Rightarrow$ L.H.S. $= \int_0^1 f((b-a)t + a)(b-a)\, dt = (b-a)\int_0^1 f\left((b-a)t + a\right)$

$= (b-a)\int_0^1 f\left((b-a)x + a\right) dx$. = RHS. Hence proved.

9. Misprinting find the sum of the integrals $\int_{-4}^{-5} e^{(x+5)^2} dx + 3\int_{1/3}^{2/3} e^{9(x-2/3)^2} dx$.

$\Rightarrow \quad I = \int_{-4}^{-5} e^{(x+5)^2} dx + \int_{1/3}^{2/3} e^{(3x-2)^2} dx$.

Replace $x + 5 = t$ in the first Integral & $3x - 2 = t$ in the second integral.

$\Rightarrow \quad I = \int_1^0 e^{t^2} dt + \int_{-1}^0 e^{t^2} dt$

In the second integral, let $t = -z \Rightarrow dt = -dz$.

$\Rightarrow \quad I = \int_1^0 e^{t^2} dt - \int_1^0 e^{z^2} dz = 0 \Rightarrow$ Sum of the given integrals = 0.

10. $\int_{-\sqrt{2}}^{\sqrt{2}}\dfrac{2x^7 - 10x^5 - 7x^3 + x}{x^2 + 2} dx + \int_{-\sqrt{2}}^{\sqrt{2}}\dfrac{3x^6 - 12x^2 + 1}{x^2 + 2} dx$

$= 0 + 2\int_0^{\sqrt{2}}\dfrac{3x^2(x^4 - 4)}{x^2 + 2} dx + 2\int_0^{\sqrt{2}}\dfrac{dx}{x^2 + 2} = 0 + 2\int_0^{\sqrt{2}} 3x^2(x^2 - 2)\, dx + 2\int_0^{\sqrt{2}}\dfrac{dx}{x^2 + 2}$

$= \dfrac{\pi}{2\sqrt{2}} - \dfrac{16\sqrt{2}}{5}$

11. (*i*) $\int_0^{\pi/2} \sin^8 x\, dx = \frac{8-1}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{105\,\pi}{8 \times 96} = \frac{35\pi}{256}$

(*ii*) $\int_0^{\pi/2} \sin^9 x \cos^7 x\, dx = \frac{(8 \cdot 6 \cdot 4 \cdot 2)\, 6 \cdot 4 \cdot 2}{16 \cdot 14 \cdot 12 \cdot 10 \cdot 8 \cdot 6 \cdot 4 \cdot 2}$.

(*iii*) $\int_0^{\pi/2} \cos^9 x\, dx = \frac{9-1}{9} \cdot \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} \cdot 1 = \frac{128}{315}$.

12. $f(x) = \sqrt{3 + x^3}$ is an inc. function in [1, 3].

$\Rightarrow$ $m = f(1) = 2$. and $M = f(3) = \sqrt{30}$. $b - a = 3 - 1 = 2$

$\Rightarrow$ $m\,(b-a) \le \int_a^b f(x)\, dx \le M\,(b-a) \quad \Rightarrow \quad 2\,(2) \le \int_1^3 \sqrt{3 + x^3}\, dx \le \left(\sqrt{30}\right)(2)$

$\Rightarrow$ $4 \le \int_1^3 \sqrt{3 + x^3}\, dx \le 2\sqrt{30}$. Hence Proved

13. (*a*) $\int_a^b e^x dx = \lim_{\substack{n \to \infty \\ h \to 0}} h \sum_{r=1}^{n} e^{a + rh} = \lim_{\substack{n \to \infty \\ h \to 0}} h \left[e^a \left(e^h + e^{2h} + e^{3h} + \ldots\ldots\ldots + e^{nh}\right)\right]$

$= \lim_{\substack{n \to \infty \\ h \to 0}} e^a h\, e^h \frac{(1 - e^{nh})}{1 - e^h} = \lim_{\substack{n \to \infty \\ h \to 0}} \frac{h}{e^h - 1} \cdot \lim_{\substack{n \to \infty \\ h \to 0}} e^h \,.\, e^a \lim_{\substack{n \to \infty \\ h \to 0}} (e^{nh} - 1)$

$= 1.1\, e^a\, (e^{b-a} - 1) \quad (\because nh = b - a)$

$= e^b - e^a$.

(*b*) $\int_a^b \sin x\, dx = \lim_{\substack{h \to 0 \\ n \to \infty}} h \sum_{r=1}^{n} \sin(a + rh) = \lim_{\substack{h \to 0 \\ n \to \infty}} h$

$\left[\sin(a+h) + \sin(a+2h) + \ldots\ldots\ldots + \sin(a+nh)\right]$

$= \lim_{\substack{h \to 0 \\ n \to \infty}} h \frac{\sin \frac{nh}{2}}{\sin \frac{h}{2}} \sin \frac{(a + h + a + nh)}{2} = \lim_{\substack{h \to 0 \\ n \to \infty}} \frac{h/2}{\sin \frac{h}{2}}\, 2 \lim_{\substack{h \to 0 \\ n \to \infty}} \sin \frac{b-a}{2} \sin \frac{(h + b + a)}{2}$

$= 1.\, 2 \sin \frac{b-a}{2} \sin \frac{b+a}{2} = \cos a - \cos b$.

14. $I = \int_{-2}^{2} \frac{dx}{x^2 + 4} = \left[\frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right)\right]_{-2}^{2} = \frac{1}{2}\left[\tan^{-1}(1) - \tan^{1}(-1)\right] = \frac{\pi}{4}$ Alternatively,

Put $x = \frac{1}{t} \quad \Rightarrow dx = \frac{-1}{t^2}\, dt$

$I = \int_{-2}^{2} \frac{dx}{4 + x^2} = \int_{-1/2}^{1/2} \frac{(-1)\, dt}{t^2\,(4 + \frac{1}{t^2})} = -\int_{-1/2}^{1/2} \frac{dt}{4t^2 + 1} = -\left[\frac{1}{2} \tan^{-1}(2t)\right]_{-1/2}^{1/2}$

$$= -\frac{1}{2}\tan^{-1}(1) - \left(-\frac{1}{2}\tan^{-1}(-1)\right) = -\frac{\pi}{4}$$

The second method is wrong because the substitution $x = \frac{1}{t}$ is discontinuous at $t = 0$. Hence substitution $x = \frac{1}{t}$ is wrong.

My Chapter Notes